MATH 426

SOLUTIONS TO HW 5

p. 107

2. $\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - + \cdots$

$$\frac{\sin z}{z^3} = z^{-2} - \frac{1}{3!} + \frac{1}{5!} z^2 - + \cdots$$

3. $$f(z) = \frac{\sin z}{\cos z} = \frac{z - z^3/3! + z^5/5! - + \cdots}{1 - \frac{z^2}{2!} + \frac{z^4}{4!} - + \cdots}$$

$$= \left(z - z^3/3! + z^5/5! - + \cdots\right) \cdot \left(1 + \left(\frac{z^2}{2!} - \frac{z^4}{4!} + \cdots\right) + \left(\frac{z^2}{2!} - \frac{z^4}{4!} + \cdots\right)^2 + \cdots\right)$$

$$= \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - + \cdots + \frac{z^3}{2!} - \frac{z^5}{3!2!} + \frac{z^7}{5!2!} - + \cdots + \cdots\right)$$

5. If $\frac{1}{f}$ has a pole at 0 Then f has a zero at 0.
If $\frac{1}{f}$ has a removable singularity at 0 Then
f has at worst a pole at 0,
So f must have an essential singularity at 0.

7. If $f(z) = \frac{1}{z}$ and $g(z) = \frac{1}{\sin z}$, then each has a pole at 0.

But $f - g = \frac{1}{z} - \frac{1}{\sin z} = \frac{\sin z - z}{z \sin z}$

has a removable singularity at 0.

However, if f, g each have a pole at 0 then $f \cdot g$ has a pole at 0.

pg. 162 - 164

1. $f(z) = \cos z \qquad f\left(\frac{\pi}{2}\right) = 0$

$f'(z) = -\sin z \qquad f'\left(\frac{\pi}{2}\right) = -1$

$f''(z) = -\cos z \qquad f''\left(\frac{\pi}{2}\right) = 0$

$f'''(z) = \sin z \qquad f'''\left(\frac{\pi}{2}\right) = 1$

$f''''(z) = \cos z \qquad f''''\left(\frac{\pi}{2}\right) = 0$

$$\therefore \cos z = -\left(z - \frac{\pi}{2}\right) + \frac{\left(z - \pi/2\right)^3}{3!} - + \cdots$$

$$\frac{1}{\cos z} = \frac{1}{-\left(z - \frac{\pi}{2}\right)} \cdot \frac{1}{1 - \frac{(z-\pi/2)^2}{3!} + \frac{(z-\pi/2)^4}{5!} - + \cdots}$$

$$= \frac{1}{-(z - \pi/2)} \left[1 + \frac{(z-\frac{\pi}{2})^2}{3!} - \frac{(z-\frac{\pi}{2})^4}{5!} + - \cdots\right]$$

$$(z + \sin z) \cdot \frac{1}{\cos z} = \frac{-1}{(z - \frac{\pi}{2})} \left[1 + \frac{(z-\frac{\pi}{2})^2}{3!} - \frac{(z-\frac{\pi}{2})^4}{5!} + - \cdots\right] \cdot$$

$$\left[\frac{\pi}{2} + (z - \tfrac{\pi}{2}) + 1 - \frac{(z-\pi/2)^2}{2!} + \frac{(z-\frac{\pi}{2})^4}{4!} - + \cdots\right]$$

$$= -\frac{1}{(z - \frac{\pi}{2})} \left[\frac{\pi}{2} + (z - \tfrac{\pi}{2}) + 1 + \frac{\frac{\pi}{2}(z-\frac{\pi}{2})^2}{3!} - \frac{(z-\frac{\pi}{2})^2}{2!} - \frac{\frac{\pi}{2}(z-\frac{\pi}{2})^4}{5!} + \frac{(z-\frac{\pi}{2})^4}{4!} + - \cdots\right]$$

$$= \frac{-\left(\frac{\pi}{2} + 1\right)}{(z - \frac{\pi}{2})} - 1 + \left(1 - \frac{\frac{\pi}{2}}{2 \cdot 3!}\right)(z - \tfrac{\pi}{2}) + \left(\frac{1}{4!} + \frac{\pi/2}{5!}\right)(z - \tfrac{\pi}{2})^3 + \cdots$$

3. $\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots$

$\sin\frac{1}{z} = \frac{1}{z} - \frac{z^{-3}}{3!} + \frac{z^{-5}}{5!} - \frac{z^{-7}}{7!} + \cdots$

9 a) pole of order 1

b) essential singularity

d) essential singularity

10. Root test:

$\left(2^{-(2^j)} z^{-j}\right)^{1/j} = \left(2^{-(2^j/j)} z^{-1}\right) \to 0 \Rightarrow j \to \infty$

So the series converges unif on cpt. sets disjoint from 0.

Since there are infinitely many terms with negative index, 0 is an essential singularity.

pg. 119–120

1. If h has a pole of order k at ∞, if $f(1/z)$ has a pole of order k at 0. If g has a pole of order k at ∞ then $g(1/z)$ has a pole of order k at 0, so $1/g(1/z)$ has a zero of order k at 0. Hence $1/g$ has a zero of order k at ∞.

4 e) $f(\frac{1}{z}) = \frac{1}{z^3} - \frac{7}{z^2} + 8$, The residue at 0 is 0,

So the residue of f at ∞ is 0.

c) $f(\frac{1}{z}) = (\frac{1}{z} + 5)^2 e^{-1/z}$

$= (\frac{1}{z^2} + \frac{10}{z} + 25)(1 - \frac{1}{z} + \frac{1}{2z^2} - + \dots)$

$= 25 + \frac{10}{z} - \frac{25}{z} - + \dots$

res at $0 = -15$

So, res of f at ∞ is -15

f) $f(\frac{1}{z}) = \frac{1}{z} - \frac{1}{3!}\frac{1}{z^3} + - \dots$

res at 0 is 1

So residue of f at ∞ is 1.