MATH 416

SOLUTIONS TO HW4

pp. 86-7

1. Let f be entire, k = positive integer.

Assume that

$$|f(z)| \leq C(1+|z|^k).$$

Look at the Cauchy estimates on a disc $D(p,R)$.

$$\left|\frac{\partial^{k+1}}{\partial z^{k+1}} f(p)\right| \leq \frac{C(1+(|p|+R)^k)(k+1)!}{R^{k+1}}$$

As $R \to \infty$ get

$$\frac{\partial^{k+1}}{\partial z^{k+1}} f \equiv 0,$$

And all higher derivatives are $\equiv 0$.

So f is a polynomial of degree k.

3. $(z-1)\cdot(z-2)\cdots(z-(m-1))\cdot(z-m)^{k-(m-1)} = p(z)$

Roots are $1, 2, 3, \ldots, m-1, m$.

9. Similar to the solution of #7

12. Let $r < \text{dist}(p, \partial U)$. Let $r < s < \text{dist}(p, \partial U)$. Then, for $z \in \overline{D}(p, r)$,

$$f(z) = \frac{1}{2\pi i} \oint_{\partial D(p,s)} \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

$$= \frac{1}{2\pi i} \oint_{\partial D(p,s)} \frac{f(\zeta)}{(\zeta - p) - (z - p)}\, d\zeta$$

$$= \frac{1}{2\pi i} \oint_{\partial D(p,s)} \frac{1}{\zeta - p} \cdot \frac{f(\zeta)}{1 - \frac{z-p}{\zeta - p}}\, d\zeta.$$

Notice that

$$\left|\frac{z-p}{s-p}\right| \leq \frac{r}{s} < 1,$$

So $$\frac{1}{1-\frac{z-p}{s-p}} = 1 + \frac{z-p}{s-p} + \left(\frac{z-p}{s-p}\right)^2 + \cdots.$$

And this series converges absolutely and uniformly on $\overline{D}(p,r)$.

So

$$f(z) = \sum_{j=0}^{\infty} \left(\frac{1}{2\pi i} \oint_{\partial D(p,s)} \frac{f(s)}{(s-p)^{j+1}} \right) \cdot (z-p)^j$$

with absolute, uniform convergence on $\overline{D}(p,r)$.

pp. 93-4

1. The set $\{x + ix^2 : x \in \mathbb{R}\}$ is a parabola. This has an interior accumulation point. By analogy with #4 on page 92, $f \equiv g$.

3. The real line has an interior accumulation point. So if f is entire and $f(x) = 0 \ \forall x \in \mathbb{R}$, then $f(z) = 0$ for all z.

But $u(x,y) = y$ is harmonic and $u(x,0) = 0 \ \forall x$.

6. $f(z) = (\sin \pi z)\cdot(\sin i\pi z)$

7. Let $|z| = 1$

$$|\varphi_c(z)| = \left|\frac{z-c}{1-\bar{c}z}\right| = \left|\frac{z-c}{1-\bar{c}z}\right| \cdot |\bar{z}|$$

$$= \left|\frac{z\cdot\bar{z} - c\bar{z}}{1-\bar{c}z}\right| = \left|\frac{1-c\bar{z}}{1-\bar{c}z}\right| . \quad (*)$$

Now the numerator is the conjugate of the denominator. So they have the same modulus. Hence $(*) = 1$.

$$\varphi_c \circ \varphi_{-c}(z) = \frac{\varphi_{-c}(z) - c}{1 - \bar{c}\varphi_{-c}(z)} = \frac{\frac{z+c}{1+\bar{c}z} - c}{1 - \bar{c}\frac{z+c}{1+\bar{c}z}}$$

$$= \frac{z+c-c(1+\bar{c}z)}{(1+\bar{c}z) - \bar{c}(z+c)} = \frac{z - |c|^2 z}{1-|c|^2} = z.$$

So φ_c is invertible of on $\overline{D}(0,1)$.
We already know that φ_c takes
$\partial D(0,2)$ to $\partial D(0,1)$. So it must
take the interior to the interior. Hence
$|z|<2 \Rightarrow |\varphi_c(z)|<1$.

$$\varphi_c(c) = \frac{c-c}{1-\bar{c}\cdot c} = 0.$$

9 Suppose f is entire and f has uncountably many zeros. Write

$$\mathbb{C} = \bigcup_{j,k=-\infty}^{\infty} \{z \in \mathbb{C} : j < \mathrm{Re}\, z \leq j+1,\ k < \mathrm{Im}\, z \leq k+1\}.$$

There are countably many of these boxes.
So some box contains uncountably many zeros.
Thus the zeros have an interior accumulation
point and $f \equiv 0$.

SOLUTIONS TO HW 4

pp. 99-100

1. a) $f(z) = \dfrac{z^2}{1-\cos z}$

At $z=0$: numerator vanishes to order 2.

Denominator $= 1-\left(1-\frac{z^2}{2}+\cdots\right) = \frac{z^2}{2}-+\cdots$

vanishes to order 2, So no singularity at $z=0$.

At $z = k\pi$, $k \neq 0$, k even, numerator does not vanish, denominator vanishes to order 2.

So f has a singularity of order 2 at $z=k\pi$, $k \neq 0$, k even.

At $z=k\pi$, k odd, no singularity.

#1 c) $h(z) = \dfrac{\sec z - 1}{\sin^2 z}$.

The denominator vanishes at $z = k\pi$.

The numerator vanishes at these points for even k.

Thus:

for k even: numerator vanishes to order 2.
denominator vanishes to order 2.
So singularity at $k\pi$ is of order 0.

for k odd: numerator does not vanish
denominator vanishes to order 2.
So singularity at $k\pi$ is of order 2.

e) $g(z) = \dfrac{z^2}{e^z - 1}$

Denominator vanishes at $z=0$ to order 1.
Numerator vanishes at $z=0$ to order 2.
So no singularity at $z=0$.

3. $f(z) = \sum_{j=-k}^{\infty} a_j (z-p)^j$

$g(z) = \sum_{k=-\infty}^{-l} b_k (z-p)^k$.

$f \cdot g = \sum_{m=-\infty}^{\infty} \sum_{j+k=m} a_j b_k (z-p)^m$

Note that, given the range of k and j, there are infinitely many terms in $f \cdot g$ with negative index. So $f \cdot g$ has an essential singularity at p.

Similar reasoning applies to $f+g$.

4. Assume f unbdd near p but $(z-p)^2 f$ is bdd near p. The second hypothesis tells us that f does not have an essential singularity at p. The first hypothesis tells us that f does not have a removable singularity at p. So f must have a pole at p of order 1 or 2.

If z is replaced by k, then f has a pole at P of order $1, 2, 3, \ldots,$ or k.