$$= \int_{-1}^{1} \frac{(1-ti)\cdot i}{1+t^2}\,dt + \int_{1}^{-1} \frac{t-i}{1+t^2}\,dt$$

$$+ \int_{1}^{-1} \frac{(-1-ti)\,i}{1+t^2}\,dt + \int_{-1}^{1} \frac{t+i}{t^2+1}\,dt$$

$$= \int_{-1}^{1} \frac{i+t}{1+t^2}\,dt + \int_{-1}^{1} \frac{-t+i}{1+t^2}\,dt$$

$$+ \int_{-1}^{1} \frac{-t+i}{1+t^2} + \int_{-1}^{1} \frac{t+i}{1+t^2}\,dt$$

$$= \int_{-1}^{1} \frac{0t+4i}{1+t^2}\,dt = 4i \arctan t \Big]_{-1}^{1}$$

$$= 4i\left(\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right) = 2i\pi.$$

7. $\oint_{\gamma} f(z)\,dz = \int_a^b f(\gamma(t))\cdot\gamma'(t)\,dt$

$$= \int_a^b F'(\gamma(t))\cdot\gamma'(t)\,dt$$

$$= F\circ\gamma(t)\Big]_a^b = F\circ\gamma(b) - F\circ\gamma(a)$$

$$= F(\gamma(b)) - F(\gamma(a)) = 0,$$

because $\gamma(a) = \gamma(b)$.

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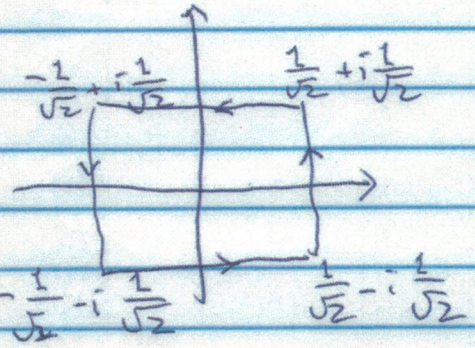
1. $\oint_\gamma F(z)\,dz = \int_0^{2\pi} \left((e^{it})^2 - e^{it}\right)\cdot ie^{it}\,dt$

$= \int_0^{2\pi} ie^{3it} - ie^{2it}\,dt$

$= \frac{e^{3it}}{3} - \frac{e^{2it}}{2}\Big]_0^{2\pi}$

$= \left(\frac{1}{3} - \frac{1}{2}\right) - \left(\frac{1}{3} - \frac{1}{2}\right) = 0.$

2.



Let

$\gamma_1(t) = \frac{1}{\sqrt{2}} + it, \; -\frac{1}{\sqrt{2}} \le t \le \frac{1}{\sqrt{2}}$

$\gamma_2(t) = t + i\frac{1}{\sqrt{2}}$, t ranges from $\frac{1}{\sqrt{2}}$ to $-\frac{1}{\sqrt{2}}$

$\gamma_3(t) = -\frac{1}{\sqrt{2}} + it$, t ranges from $\frac{1}{\sqrt{2}}$ to $-\frac{1}{\sqrt{2}}$

$\gamma_4(t) = t - i\frac{1}{\sqrt{2}}, \; -\frac{1}{\sqrt{2}} \le t \le \frac{1}{\sqrt{2}}$

$\oint_{\gamma_1} F(z)\,dz = \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} \left[\left(\frac{1}{\sqrt{2}} + it\right)^2 - \left(\frac{1}{\sqrt{2}} + it\right)\right]\cdot i\,dt$

$= i\int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} \frac{1}{2} + \sqrt{2}it - t^2 - \frac{1}{\sqrt{2}} - it\,dt$

$$= i\left(\frac{t}{2} + \frac{\sqrt{2}}{2} i t^2 - \frac{t^3}{3} - \frac{t}{\sqrt{2}} - \frac{i t^2}{2}\right)\Bigg]_{\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}}$$

$$= i\left(\frac{1}{2\sqrt{2}} + \frac{\sqrt{2}}{4} i - \frac{1}{3\cdot 2\sqrt{2}} - \frac{1}{2} - \frac{i}{4}\right)$$

$$- i\left(-\frac{1}{2\sqrt{2}} + \frac{\sqrt{2}}{4} i + \frac{1}{3\cdot 2\sqrt{2}} + \frac{1}{2} - \frac{i}{4}\right)$$

$$= \frac{i}{\sqrt{2}} - \frac{1}{3\sqrt{2}} - i = \frac{3i}{3\sqrt{2}} - \frac{i}{3\sqrt{2}} - \frac{3\sqrt{2}\, i}{3\sqrt{2}} = \frac{2i - 3\sqrt{2}\, i}{3\sqrt{2}}$$

$$\oint_{\gamma_2} f(z)\,dz = \int_{\frac{1}{\sqrt{2}}}^{-\frac{1}{\sqrt{2}}} \left[\left(t + i\frac{1}{\sqrt{2}}\right)^2 - \left(t + i\frac{1}{\sqrt{2}}\right)\right] \cdot 1\, dt$$

$$= \int_{\frac{1}{\sqrt{2}}}^{-\frac{1}{\sqrt{2}}} t^2 + \sqrt{2}\, i t - \frac{1}{2} - t - \frac{i}{\sqrt{2}}\, dt$$

$$= \frac{t^3}{3} + \frac{\sqrt{2}}{2} i t^2 - \frac{1}{2} t - \frac{t^2}{2} - \frac{i t}{\sqrt{2}}\Bigg]_{\frac{1}{\sqrt{2}}}^{-\frac{1}{\sqrt{2}}}$$

$$= \left(-\frac{1}{3\cdot 2\sqrt{2}} + \frac{\sqrt{2}}{4} i + \frac{1}{2\sqrt{2}} - \frac{1}{4} + \frac{i}{2}\right)$$

$$- \left(\frac{1}{3\cdot 2\sqrt{2}} + \frac{\sqrt{2}}{4} i - \frac{1}{2\sqrt{2}} - \frac{1}{4} - \frac{i}{2}\right)$$

$$= -\frac{1}{3\sqrt{2}} + i + \frac{1}{\sqrt{2}} = \frac{+1 + 3\sqrt{2}\, i}{3\sqrt{2}}$$

$$\oint_{\gamma_3} f(z)\,dz = \int_{\frac{1}{\sqrt{2}}}^{-\frac{1}{\sqrt{2}}} \left[\left(-\frac{1}{\sqrt{2}} + it\right)^2 - \left(-\frac{1}{\sqrt{2}} + it\right)\right] \cdot i\, dt$$

$$= i \int_{\frac{1}{\sqrt{2}}}^{-\frac{1}{\sqrt{2}}} \frac{1}{2} - \sqrt{2}\, i t - t^2 + \frac{1}{\sqrt{2}} - it\, dt$$

$$= i\left(\frac{t}{2} - \frac{\sqrt{2}}{2} i t^2 - \frac{t^3}{3} + \frac{t}{\sqrt{2}} - i\frac{t^2}{2}\right]_{1/\sqrt{2}}^{-1/\sqrt{2}}$$

$$= i\left(-\frac{1}{2\sqrt{2}} - \frac{\sqrt{2}}{4}i + \frac{1}{3\cdot 2\sqrt{2}} - \frac{1}{2} - \frac{i}{4}\right)$$

$$- i\left(\frac{1}{2\sqrt{2}} - \frac{\sqrt{2}}{4}i - \frac{1}{3\cdot 2\sqrt{2}} + \frac{1}{2} - \frac{i}{4}\right)$$

$$= -i\cdot\frac{1}{\sqrt{2}} + \frac{i}{3\sqrt{2}} - i = \frac{-3i + i - 3\sqrt{2}i}{3\sqrt{2}} = \frac{-2i - 3\sqrt{2}i}{3\sqrt{2}}$$

$$\oint_{\gamma_4} f(z)\,dz = \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} \left[\left(t - i\tfrac{1}{\sqrt{2}}\right)^2 - \left(t - i\tfrac{1}{\sqrt{2}}\right)\right]\cdot 1\,dt$$

$$= \int_{-1/\sqrt{2}}^{1/\sqrt{2}} t^2 - \sqrt{2}\,it - \frac{1}{2} - t + i\cdot\frac{1}{\sqrt{2}}\,dt$$

$$= \frac{t^3}{3} - \frac{\sqrt{2}}{2}it^2 - \frac{t}{2} - \frac{t^2}{2} + \frac{i}{\sqrt{2}}t \Bigg]_{-1/\sqrt{2}}^{1/\sqrt{2}}$$

$$= \left(\frac{1}{3\cdot 2\sqrt{2}} - \frac{\sqrt{2}}{4}i - \frac{1}{2\sqrt{2}} - \frac{1}{4} + \frac{i}{2}\right)$$

$$- \left(-\frac{1}{3\cdot 2\sqrt{2}} - \frac{\sqrt{2}}{4}i + \frac{1}{2\sqrt{2}} - \frac{1}{4} - \frac{i}{2}\right)$$

$$= \frac{1}{3\sqrt{2}} - \frac{1}{\sqrt{2}} + i = \frac{1 - 3 + 3\sqrt{2}i}{3\sqrt{2}} = \frac{-2 + 3\sqrt{2}i}{3\sqrt{2}}$$

Then

$$\oint_{\gamma_1} f(z)\,dz + \oint_{\gamma_2} f(z)\,dz + \oint_{\gamma_3} f(z)\,dz + \oint_{\gamma_4} f(z)\,dz = 0,$$

6. If γ_1 can be continuously deformed to γ_2, and if $\tilde{\gamma}_2$ is γ_2 with reverse orientation, then γ_1 followed by $\tilde{\gamma}_2$ is a closed curve that is homotopic to 0. So

$$\oint_{\gamma_1} f(z)\,dz + \oint_{\tilde{\gamma}_2} f(z)\,dz = 0$$

hence

$$\oint_{\gamma_1} f(z)\,dz = -\oint_{\tilde{\gamma}_2} f(z)\,dz$$

$$= \oint_{\gamma_2} f(z)\,dz,$$

11. $$\frac{1}{2\pi i}\oint_{\gamma} \frac{f(\zeta)}{\zeta - p}\,d\zeta = \frac{1}{2\pi i}\oint \frac{1/\zeta}{\zeta - 2}\,d\zeta$$

$$= \frac{1}{2\pi i}\oint_{\gamma} \frac{1}{\zeta}\cdot\frac{1}{\zeta}\cdot\frac{1}{1-\frac{2}{\zeta}}\,d\zeta$$

$$= \frac{1}{2\pi i}\oint_{\gamma} \frac{1}{\zeta^2}\cdot\left(1+\frac{2}{\zeta}+\frac{4}{\zeta^2}+\frac{8}{\zeta^3}+\cdots\right)d\zeta$$

$$= \frac{1}{2\pi i}\oint_{\gamma} \frac{1}{\zeta^2}+\frac{2}{\zeta^3}+\frac{4}{\zeta^4}+\frac{8}{\zeta^5}+\cdots\,d\zeta$$

$$= \frac{1}{2\pi i}\int_0^{2\pi}\left(\frac{1}{9e^{2it}}+\frac{2}{27e^{3it}}+\frac{4}{81e^{4it}}+\frac{8}{243e^{5it}}+\cdots\right)3ie^{it}\,dt$$

$$= \frac{1}{2\pi i} \int_0^{2\pi} \frac{3}{9e^{it}} + \frac{6}{27e^{2it}} + \frac{12}{81e^{3it}} + \frac{24}{243e^{4it}} + \cdots \, dt$$

$$= 0 \neq \frac{1}{2} = f(p).$$

13. $\frac{1}{2\pi i} \oint_\gamma \frac{1}{s - z} \, ds$

$$= \frac{1}{2\pi i} \oint_\gamma \frac{1}{s} \cdot \frac{1}{1 - \frac{z}{s}} \, ds$$

$$= \frac{1}{2\pi i} \oint_\gamma \frac{1}{s} \cdot \left(1 + \frac{z}{s} + \frac{z^2}{s^2} + \cdots\right) ds$$

$$= \frac{1}{2\pi i} \int_0^{2\pi} \left(\frac{1}{e^{it}} + \frac{z}{e^{2it}} + \frac{z^2}{e^{3it}}\right) ie^{it} \, dt$$

$$= \frac{1}{2\pi} \int_0^{2\pi} 1 + ze^{-it} + z^2 e^{-2it} \, dt$$

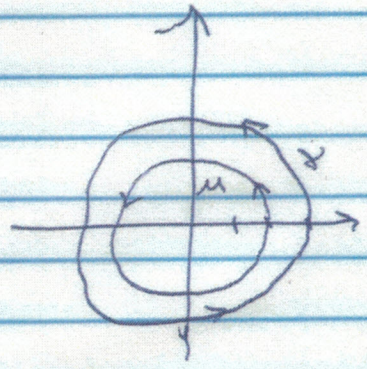
$$= \frac{1}{2\pi} \int_0^{2\pi} 1 \, dt = 1.$$

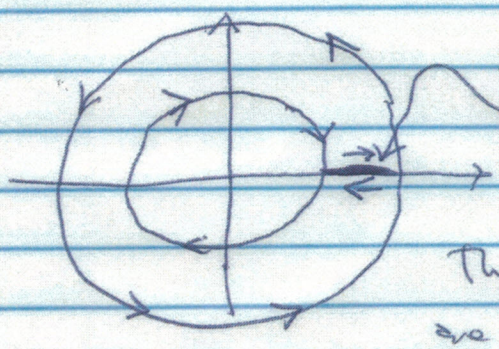
Also the Cauchy integral formula applied to the holomorphic function $f(z) = 1$ gives

$$1 = f(z) = \frac{1}{2\pi i} \oint_\gamma \frac{f(s)}{s - z} \, ds.$$

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1.





By our usual trick of interpolating a segment here we see that the two integrals are equal.

6. $\frac{1}{2\pi i}\oint_{\partial D(0,1)} \bar{\zeta}^k \, d\zeta = \frac{1}{2\pi i}\int_0^{2\pi} e^{-ikt} \, ie^{it} \, dt$

$= \frac{1}{2\pi}\int_0^{2\pi} e^{-i(k-1)t} \, dt$

$= -\frac{1}{2\pi i}\frac{e^{-i(k-1)t}}{k-1}\Big|_0^{2\pi}$

$= 0$ if $k \neq 1$

If $k = 1$ then this equals 1.

$k<0$

8. $\frac{1}{2\pi i}\oint_{\partial D(0,1)} z^k\,dz = \frac{1}{2\pi i}\int_0^{2\pi} e^{ikt}\, ie^{it}\,dt$

$$= \frac{1}{2\pi}\int_0^{2\pi} e^{i(k+1)t}\,dt$$

$$= \frac{1}{2\pi i}\,\frac{e^{i(k+1)t}}{k+1}\Big]_0^{2\pi}$$

$$= 0 \quad \text{if } k \neq -1,$$

In case $k=-1$ get

$$\frac{1}{2\pi}\int_0^{2\pi} 1\,dt = 1,$$