Math 416

Solutions to HW 3

pages 52-53

1a) $f(z) = z\bar{z}^2 - \cos z$, $\gamma(t) = \cos 2t + i \sin 2t$, $0 \le t \le \frac{\pi}{2}$

$$\oint_\gamma f\,dz = \int_0^{\frac{\pi}{2}} \Big[(\cos 2t + i\sin 2t)\cdot(\cos 2t - i\sin 2t)^2 - \cos(\cos 2t + i\sin 2t) \Big] \cdot (-2\sin 2t + 2i\cos 2t)\,dt$$

$$= \int_0^{\frac{\pi}{2}} \Big[(\cos 2t + i\sin 2t)(\cos^2 2t - \sin^2 2t - 2i\cos 2t \sin 2t) - \cos(\cos 2t + i\sin 2t) \Big] \cdot (-2\sin 2t + 2i\cos 2t)\,dt$$

$$= \int_0^{\frac{\pi}{2}} \Big[\cos^3 2t - \cos 2t \sin^2 2t - 2i \cos^2 2t \sin 2t + i \sin 2t \cos^2 2t - i \sin^3 2t + 2 \cos 2t \sin^2 2t) - \cos(\cos 2t + i \sin 2t) \Big] \cdot (-2 \sin 2t + 2i \cos 2t)\, dt$$

$$= \int_0^{\frac{\pi}{2}} -2\cos^3 2t \sin 2t + 2 \cos 2t \sin^3 2t + 4i \cos^2 2t \sin^2 2t - 2i \sin^2 2t \cos^2 2t + 2i \sin^4 2t - 4 \cos 2t \sin^3 2t + 2i \cos^4 2t - 2i \cos^2 2t \sin^2 2t + 4 \cos^3 2t \sin 2t - 2 \sin 2t \cos^3 2t + 2 \sin^3 2t \cos 2t + 4i \cos^2 2t \sin^2 2t + 2 \cos(\cos 2t + i \sin 2t) \sin 2t - 2i \cos(\cos 2t + i \sin 2t) \cos 2t \, dt$$

$$= \int_0^{\pi/2} 4i \cos^2 2t \sin^2 2t + 2i \sin^4 2t + 2i \cos^4 2t + 2 \cos(\cos 2t + i \sin 2t) \sin 2t - 2i \cos(\cos 2t + i \sin 2t) \cos 2t \, dt \qquad (*)$$

Now $\cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha = \cos^2 \alpha - (1 - \cos^2 \alpha)$

$= 2\cos^2 \alpha - 1$

So $\cos^2 \alpha = \dfrac{1 + \cos 2\alpha}{2}$

Similarly, $\sin^2 \alpha = \dfrac{1 - \cos 2\alpha}{2}$.

Thus

$$4i\cos^2 2t\sin^2 2t = i(2\cos 2t \sin 2t)^2 = i\sin^2 4t$$

$$= i\cdot\frac{1-\cos 8t}{2}$$

$$2i\sin^4 2t = 2i(\sin^2 2t)(\sin^2 2t)$$

$$= 2i\left(\frac{1-\cos 4t}{2}\right)\cdot\left(\frac{1-\cos 4t}{2}\right)$$

$$= 2i\left(\frac{1}{4} - \frac{\cos 4t}{2} + \frac{\cos^2 4t}{4}\right)$$

$$= \frac{i}{2} - i\cos 4t + i\left(\frac{1+\cos 8t}{4}\right)$$

$$2i\cos^4 2t = 2i(\cos^2 2t)(\cos^2 2t)$$

$$= 2i\left(\frac{1+\cos 4t}{2}\right)\left(\frac{1+\cos 4t}{2}\right)$$

$$= \frac{i}{2} + i\cos 4t + \frac{i\cos^2 4t}{2}$$

$$= \frac{i}{2} + i\cos 4t + \frac{i(1+\cos 8t)}{4}$$

Thus our integral (*) is

$$= \int_0^{\pi/2} i\cdot\frac{1-\cos 8t}{2} + \left(\frac{i}{2} - i\cos 4t + i\left(\frac{1+\cos 8t}{4}\right)\right)$$

$$+ \left(\frac{i}{2} + i\cos 4t + \frac{i(1+\cos 8t)}{4}\right)$$

$$+ 2\cos(\cos 2t + i\sin 2t)\sin 2t$$

$$- 2i\cos(\cos 2t + i\sin 2t)\cos 2t\, dt$$

$$= i\frac{t}{2} - i\frac{\sin 8t}{16} + \frac{i}{2}t - i\frac{\sin 4t}{4} + \frac{i}{4}t + i\frac{\sin 8t}{32}$$
$$+ \frac{it}{2} + i\frac{\sin 4t}{4} + \frac{i}{4}t + \frac{i\sin 8t}{32}\Big]_0^{\pi/2}$$

$$+ \int_0^{\pi/2} 2\cos(\cos 2t + i\sin 2t)\sin 2t - 2i\cos(\cos 2t + i\sin 2t) \cdot \cos 2t\, dt$$

$$= \left(i\frac{\pi}{4} + 0 + i\frac{\pi}{4} + 0 + i\frac{\pi}{8} + 0 + i\frac{\pi}{4} + 0 + i\frac{\pi}{8} + 0\right)$$
$$-(0 - 0 + 0 - 0 + 0 + 0 + 0 + 0 + 0 + 0)$$

$$+ \int_0^{\pi/2} 2\cos(\cos 2t + i\sin 2t)\sin 2t - 2i\cos(\cos 2t + i\sin 2t) \cdot \cos 2t\, dt$$

$$= \pi i + \int_0^{\pi/2} 2\cos(\cos 2t + i\sin 2t)\sin 2t$$
$$- 2i\cos(\cos 2t + i\sin 2t)\cos 2t\, dt$$

c) $f(z) = z^3 + \frac{z}{z+1}$, $\gamma(t) = e^t + ie^{2t}$, $1 \le t \le 2$

$$\oint_\gamma f(z)\,dz = \int_1^2 \left[\left(e^t + ie^{2t}\right)^3 + \frac{e^t + ie^{2t}}{e^t + ie^{2t} + 1} \right] \cdot \left(e^t + 2ie^{2t}\right) dt$$

$$= \int_1^2 \left(e^{3t} + 3ie^{4t} - 3e^{5t} - ie^{6t}\right)\left(e^t + 2ie^{2t}\right) + \frac{e^{2t} + ie^{3t} + 2ie^{3t} - 2e^{4t}}{e^t + ie^{2t} + 1}\,dt$$

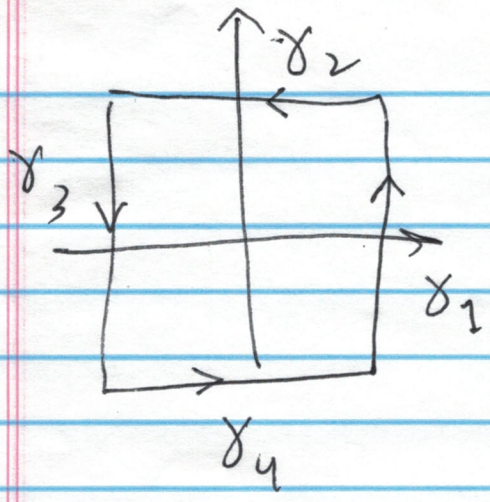
$$= \int_1^2 \left(e^{4t} + 2ie^{5t} + 3ie^{5t} - 6e^{6t} - 3e^{6t} - 6ie^{7t} - ie^{7t} + 2e^{8t}\right) dt + \int_1^2 \frac{e^{2t} + ie^{3t} + 2ie^{3t} - 2e^{4t}}{e^t + ie^{2t} + 1}\,dt$$

$$= \int_1^2 \left(e^{4t} + 5ie^{5t} - 9e^{6t} - 7ie^{7t} + 2e^{8t}\right) dt + \int_1^2 \frac{e^{2t} + 3ie^{3t} - 2e^{4t}}{e^t + ie^{2t} + 1}\,dt$$

The first integral is easy. The second is not.

(6)

2.



$\gamma = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$

$\gamma_1(t) = 1 + ti$, $-1 \le t \le 1$

$\gamma_2(t) = t + i$, t ranges from 1 to -1

$\gamma_3(t) = -1 + ti$, t ranges from 1 to -1

$\gamma_4(t) = t - i$, $-1 \le t \le 1$

So $$\oint_\gamma f\,dz = \int_{-1}^{1} (1+ti)^2 \cdot i\,dt + \int_{1}^{-1} (t+i)^2 \cdot 1\,dt$$
$$+ \int_{1}^{-1} (-1+ti)^2 \cdot i\,dt + \int_{-1}^{1} (t-i)^2 \cdot 1\,dt$$

$$= \int_{-1}^{1} i - it^2 - 2t\,dt + \int_{1}^{-1} t^2 - 1 + 2it\,dt$$
$$+ \int_{1}^{-1} i - it^2 + 2t\,dt + \int_{-1}^{1} t^2 - 1 - 2it\,dt$$

$$= it - \frac{it^3}{3} - t^2 \Big]_{-1}^{1} + \frac{t^3}{3} - t + it^2 \Big]_{1}^{-1}$$
$$+ it - \frac{it^3}{3} + t^2 \Big]_{1}^{-1} + \frac{t^3}{3} - t + it^2 \Big]_{-1}^{1}$$

$$= \left(\left(i - \frac{i}{3} - 1\right) - \left(-1 + \frac{i}{3} - 1\right)\right) + \left(\left(-\frac{1}{3} + 1 + i\right) - \left(\frac{1}{3} - 1 + i\right)\right)$$

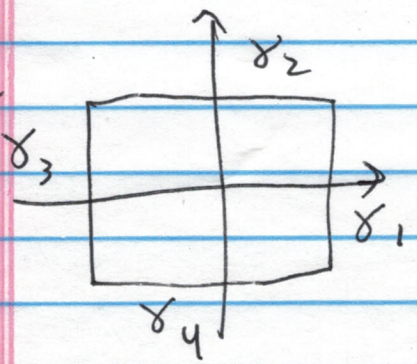
$$+ \left(\left(-1 + \frac{i}{3} + 1\right) - \left(i - \frac{i}{3} + 1\right)\right) + \left(\left(\frac{1}{3} - 1 + i\right) - \left(-\frac{1}{3} + 1 + i\right)\right)$$

$$= \left(\frac{2}{3}i - 1 + \frac{2}{3}i + 1\right) + \left(\frac{2}{3} + i + \frac{2}{3} - i\right)$$

$$+ \left(-\frac{2}{3}i + 1 - \frac{2i}{3} - 1\right) + \left(-\frac{2}{3} + i - \frac{2}{3} - i\right)$$

$$= 0 + 0i$$

3,



$\gamma_1(t) = 1 + ti$, $-1 \le t \le 1$

$\gamma_2(t) = t + i$, t ranges from 1 to -1

$\gamma_3(t) = -1 + ti$, t ranges from 1 to -1

$\gamma_4(t) = t - i$, , $-1 \le t \le 1$

$$\oint_\gamma f(z)\,dz = \int_{-1}^{1} \frac{1}{1+ti} \cdot i\,dt + \int_{1}^{-1} \frac{1}{t+i} \cdot 1\,dt$$

$$+ \int_{1}^{-1} \frac{1}{-1+ti} \cdot i\,dt + \int_{-1}^{1} \frac{1}{t-i} \cdot 1\,dt$$