# MATH 416 HOMEWORK 2
## SOLUTIONS

p. 26

1. $i = 1 \cdot e^{i\frac{\pi}{2}}$

$(re^{i\theta})^3 = 1 \cdot e^{i\frac{\pi}{2}} \Rightarrow r = 1, \theta = \frac{\pi}{6}.$

$root_1 = 1 \cdot e^{i\frac{\pi}{6}} = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6} = \frac{\sqrt{3}}{2} + i\frac{1}{2}.$

$(re^{i\theta})^3 = 1 \cdot e^{i\frac{5\pi}{2}} \Rightarrow r = 1, \theta = \frac{5\pi}{6}$

$root_2 = 1 \cdot e^{i\frac{5\pi}{6}} = \cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6} = -\frac{\sqrt{3}}{2} + i\frac{1}{2}$

$(re^{i\theta})^3 = 1 \cdot e^{i\frac{9\pi}{2}} \Rightarrow r = 1, \theta = \frac{9\pi}{6} = \frac{3\pi}{2}$

$root_3 = 1 \cdot e^{i\frac{3\pi}{2}} = \cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2} = 0 - i.$

4. $-1 + i = \sqrt{2} \cdot e^{i\frac{3\pi}{4}}$

$(re^{i\theta})^5 = \sqrt{2} \cdot e^{i\frac{3\pi}{4}} \Rightarrow r = 2^{1/10}, \theta = \frac{3\pi}{20}$

$root_1 = 2^{1/10}\left(\cos\frac{3\pi}{20} + i\sin\frac{3\pi}{20}\right)$

$(re^{i\theta})^5 = \sqrt{2} \cdot e^{i\frac{11\pi}{4}} \Rightarrow r = 2^{1/10}, \theta = \frac{11\pi}{20}$

$root_2 = 2^{1/10}\left(\cos\frac{11\pi}{20} + i\sin\frac{11\pi}{20}\right)$

$(re^{i\theta})^5 = \sqrt{2} \cdot e^{i\frac{19\pi}{4}} \Rightarrow r = 2^{1/10}, \theta = \frac{19\pi}{20}$

$root_3 = 2^{1/10}\left(\cos\frac{19\pi}{20} + i\sin\frac{19\pi}{20}\right)$

$(re^{i\theta})^5 = \sqrt{2} \cdot e^{i\frac{27\pi}{4}} \Rightarrow r = 2^{1/10}, \theta = \frac{27\pi}{20}$

$root_4 = 2^{1/10}\left(\cos\frac{27\pi}{20} + i\sin\frac{27\pi}{20}\right)$

$(re^{i\theta})^5 = \sqrt{2} \cdot e^{i\frac{35\pi}{4}} \Rightarrow r = 2^{1/10}, \theta = \frac{35\pi}{20}$

$root_5 = 2^{1/10}\left(\cos\frac{35\pi}{20} + i\sin\frac{35\pi}{20}\right)$

6. $|z| \le |\mathrm{Re}\, z| + |\mathrm{Im}\, z|$

$\sqrt{(\mathrm{Re}\, z)^2 + (\mathrm{Im}\, z)^2} \le |\mathrm{Re}\, z| + |\mathrm{Im}\, z|$

$(\mathrm{Re}\, z)^2 + (\mathrm{Im}\, z)^2 \le (\mathrm{Re}\, z)^2 + (\mathrm{Im}\, z)^2 + 2|\mathrm{Re}\, z||\mathrm{Im}\, z|$

This last is true, so first line is true as well

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p. 34

2b) $g(z) = e^{2z - z^3} - z^2$

$\frac{\partial g}{\partial \bar{z}} = \frac{\partial}{\partial \bar{z}}\left(e^{2z - z^3}\right) - \frac{\partial}{\partial \bar{z}} z^2$

$= e^{2z - z^3} \cdot \frac{\partial}{\partial \bar{z}}(2z - z^3) - \frac{\partial}{\partial \bar{z}} z^2 = 0.$

d) $k(z) = z(\sin z + z)$.

$\frac{\partial k}{\partial \bar{z}} = \frac{\partial}{\partial \bar{z}}\left[z(\sin z + z)\right]$

$= \frac{\partial z}{\partial \bar{z}} \cdot (\sin z + z) + z \cdot \frac{\partial}{\partial \bar{z}}(\sin z + z)$

$= 0 + 0 = 0$

2b) $g(z) = \frac{\bar{z}}{z^2 + 1}$

$\frac{\partial g}{\partial \bar{z}} = \frac{(z^2+1)\frac{\partial \bar{z}}{\partial \bar{z}} - \bar{z}\frac{\partial}{\partial \bar{z}}(z^2+1)}{(z^2+1)^2}$

$= \frac{(z^2+1)\cdot 1 - 0}{(z^2+1)^2} = \frac{1}{(z^2+1)} \neq 0.$

So not holomorphic

c) $\frac{\partial h}{\partial \bar{z}} = \frac{\partial}{\partial \bar{z}}\left(z \cdot (\bar{z}^2 - z)\right)$

$= \frac{\partial z}{\partial \bar{z}} \cdot (\bar{z}^2 - z) + z \cdot \frac{\partial}{\partial \bar{z}}(\bar{z}^2 - z)$

$= 0 + z \cdot 2\bar{z} \neq 0.$

3 b) $\frac{\partial}{\partial z}\left[\cos z (1 + \sin^2 z)\right]$

$= \frac{\partial \cos z}{\partial z} \cdot (1 + \sin^2 z) + \cos z \cdot \frac{\partial}{\partial z}(1 + \sin^2 z)$

$= -\sin z \cdot (1 + \sin^2 z) + \cos z \cdot 2 \sin z \cdot \cos z.$

4 c) $\frac{\partial}{\partial \bar{z}}\left[\cos z (1 + z \cos \bar{z})\right]$

$= \frac{\partial \cos z}{\partial \bar{z}} \cdot (1 + z \cos \bar{z}) + \cos z \cdot \frac{\partial}{\partial \bar{z}}(1 + z \cos \bar{z})$

$= 0 + \cos z \cdot z \cdot \frac{\partial}{\partial \bar{z}} \cos \bar{z}$

$= z \cos z \cdot (-\sin \bar{z})$

8. Antidifferentiate $\sin z$. So

$g(z) = \frac{z^2}{2} \cdot \bar{z} + \cos z$

and $\frac{\partial g}{\partial z} = z \cdot \bar{z} - \sin z$

16. Anti differentiate twice. So

$$k(z) = \frac{|z|^4}{4} + \bar{z}\cos z + \frac{z\bar{z}^{-4}}{4} = \frac{z^2\bar{z}^{-2}}{4} + \bar{z}\cos z + \frac{z\bar{z}^{-4}}{4}.$$

Then $\frac{\partial k}{\partial z} = \frac{2z\bar{z}^2}{4} - \bar{z}\sin z + \frac{\bar{z}^4}{4}$

$$\frac{\partial^2 k}{\partial z \partial \bar{z}} = \frac{2z2\bar{z}}{4} - \sin z + \frac{4\bar{z}^3}{4}$$

$$= z\bar{z} - \sin z + \bar{z}^3$$

$$= |z|^2 - \sin z + \bar{z}^3$$

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2a) $f(z) = \operatorname{Re} z = \frac{z+\bar{z}}{2}$

$$\Delta f = 4\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}}\left(\frac{z+\bar{z}}{2}\right) = 4\frac{\partial}{\partial z}\left(\frac{1}{2}\right) = 0$$

c) $h(z) = |z|^2 - 2x^2 = y^2 - x^2$

$$\Delta h = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(y^2 - x^2) = 2 - 2 = 0$$

2b) $g(z) = |z|^4 - |z|^2$

$$4\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}} g = 4(4|z|^2 - 1) \neq 0$$

d) $k(z) = e^{\bar{z}\cos z}$

$$4\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}} k = 4\frac{\partial}{\partial z}\left(\cos z\, e^{\bar{z}\cos z}\right) = 4\left(-\sin z\, e^{\bar{z}\cos z}\right.$$
$$\left. + \cos z\left(\bar{z}(-\sin z)\right) e^{\bar{z}\cos z}\right) \neq 0$$

3 b) $u(z) = e^x \sin y$

$$e^x \sin y = \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \Rightarrow v = -e^x \cos y + \varphi(x)$$

$$e^x \cos y = \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$-\frac{\partial v}{\partial x} = -\left(-e^x \cos y + \varphi'(x)\right) = e^x \cos y$$

$$\therefore \varphi'(x) = 0$$

$$\varphi(x) = C$$

So $v(x) = -e^x \cos y + C$

d) $u(z) = x - y$

$$1 = \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \Rightarrow v = y + \varphi(x)$$

$$-1 = \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$-\frac{\partial v}{\partial x} = -\varphi'(x) = -1 \Rightarrow \varphi(x) = +x + C.$$

So $v = y + x + C$

4. Of course $r^2 = x^2+y^2$, $x = r\cos\theta$, $y = r\sin\theta$.

So
$$\frac{\partial}{\partial r} = \frac{\partial x}{\partial r}\frac{\partial}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial}{\partial y} = \cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y}$$
$$\frac{\partial}{\partial \theta} = \frac{\partial x}{\partial \theta}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \theta}\frac{\partial}{\partial y} = -r\sin\theta\frac{\partial}{\partial x} + r\cos\theta\frac{\partial}{\partial y}.$$

We solve these equations for $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$. Thus

$$\frac{\partial}{\partial x} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}$$
$$\frac{\partial}{\partial y} = \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}$$

Now we see that

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)\left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)$$
$$+\left(\sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\right)\left(\sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\right)$$
$$= \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}.$$

6. $f(z) = z^2\cos z - e^{z^3 - z}$

$$= z^2\frac{e^{iz}+e^{-iz}}{2} - e^{z^3}e^{-z}$$
$$= ((x^2-y^2)+2ixy)\left(\frac{e^{ix-y}+e^{-ix+y}}{2}\right) - e^{(x^3-3xy^2)+i(3yx^2-y^3)}e^{-x-iy}$$
$$= ((x^2-y^2)+2ixy)\left(\frac{e^{-y}(\cos x + i\sin x) + e^{y}(\cos x - i\sin x)}{2}\right)$$
$$- e^{(x^3-3xy^2)}e^{-x}\left(\cos(3yx^2-y^3) + i\sin(3yx^2-y^3)\right)(\cos y - i\sin y)$$

And the real and imaginary parts can be worked out from this just with algebra.

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1. $f(z) = z^2 = (x^2 - y^2) + 2ixy$

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

$$= \lim_{h \to 0} \frac{(z+h)^2 - z^2}{h} = \lim_{h \to 0} \frac{z^2 + 2hz + h^2 - z^2}{h}$$

$$= \lim_{h \to 0} \frac{2hz + h^2}{h} = \lim_{h \to 0} 2z + h = 2z.$$

$$J(p) = \begin{pmatrix} u_x(p) & u_y(p) \\ -u_y(p) & u_x(p) \end{pmatrix}$$

$$= \begin{pmatrix} 2x & -2y \\ 2y & 2x \end{pmatrix}$$

$$= \sqrt{4x^2+4y^2} \begin{pmatrix} \frac{2x}{\sqrt{4x^2+4y^2}} & \frac{-2y}{\sqrt{4x^2+4y^2}} \\ \frac{2y}{\sqrt{4x^2+4y^2}} & \frac{2x}{\sqrt{4x^2+4y^2}} \end{pmatrix}$$

Each row of the matrix has norm 1, and the rows are orthogonal, The dilation factor is $\sqrt{4x^2+4y^2}$, The determinant is +1.

5. $g(z) = \bar{z}^2 = (x^2 - y^2) - 2ixy$

$$J(P) = \begin{pmatrix} u_x(P) & u_y(P) \\ v_x(P) & v_y(P) \end{pmatrix}$$

$$= \begin{pmatrix} 2x & -2y \\ -2y & -2x \end{pmatrix}$$

$$= \sqrt{4x^2+4y^2} \cdot \begin{pmatrix} \frac{2x}{\sqrt{4x^2+4y^2}} & \frac{-2y}{\sqrt{4x^2+4y^2}} \\ \frac{-2y}{\sqrt{4x^2+4y^2}} & \frac{-2x}{\sqrt{4x^2+4y^2}} \end{pmatrix}$$

The matrix has rows of norm 1. And they are orthogonal. But the determinant is $-1$. So angles are reversed.

6. $h(z) = z + 2\bar{z} = (z+\bar{z}) + \bar{z} = 2x + (x - iy) = 3x - iy$

$$J(P) = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= 3 \begin{pmatrix} \frac{3}{3} & 0 \\ 0 & -\frac{1}{3} \end{pmatrix} = 3 \begin{pmatrix} 1 & 0 \\ 0 & -\frac{1}{3} \end{pmatrix}$$

The rows are orthogonal but do not have norm 1. The determinant is negative.
So <u>not</u> conformal.