Solutions to HW 6

Ch. 8 #2. Let $E \in \mathcal{X}$. Then $\lambda(P_1 \cap E) \geq 0$ and $\lambda(P_2 \cap E) \geq 0$. Then

$$\lambda((P_1 \cup P_2) \cap E) = \lambda(P_1 \cap E) + \lambda(P_2 \cap (E \cap {}^c P_1)) \geq 0.$$

3. If $\mu_1 << \mu_2$ and $\mu_2 << \mu_3$, Then let $\mu_3(E) = 0$. Then $\mu_2(E) = 0$. So $\mu_1(E) = 0$. Hence $\mu_1 << \mu_3$.

6. First check for g a characteristic function, then for g a simple function. Finally for $g \geq 0$ a measurable function.

7. If $\lambda \perp \mu$ then $X = A \cup B$ with $\lambda(A) = 0$, $\mu(B) = 0$. But $\mu(B) = 0 \Rightarrow \lambda(B) = 0$ since $\lambda << \mu$. Hence $\lambda(X) = \lambda(A \cup B) = \lambda(A) + \lambda(B) = 0$.

Ch. 9 # 1. It is not an algebra because it does not respect complementation. The σ algebra containing $\mathcal{G}$ will contain all bounded intervals —

open, closed, and half-open. So it will be the Borel sets.

3. Consider $[0,1) \setminus \left([0,\tfrac{1}{2}) \cup (\tfrac{1}{2},1]\right) = \{\tfrac{1}{2}\}$.
This set is not in $\mathcal{A}$. So $\mathcal{A}$ is not an algebra.
This example also shows that not every nonempty set is infinite.

6. We have in the past constructed a Cantor-like set with positive measure (by excising intervals of lengths $1/5^k$). The interior of this set is empty, so has measure 0.

20. We know that we can approximate g in the L^1 topology by a simple function s. Say that $s = \sum a_j \chi_{E_j}$. Using the concept of outer measure, we can approximate each χ_{E_j} by the characteristic function of an open set $\mathcal{O}_j$. Then, as we have done in class, we can approximate each $\chi_{\mathcal{O}_j}$ by a continuous function.

Ch. 10 #2. If φ is a simple function then it is clear. Any Lebesgue integrable φ can be approximated by simple functions.

If $\tau_a f(x) = f(x+a)$ for $a \in \mathbb{R}$, then notice that

$$T(\tau_a f) = \int f((x-t)+a)\,\varphi(t)\,dt$$

$$= \int f((x+a)-t)\,\varphi(t)\,dt$$

$$= \tau_a T f.$$

5. We know that there is a set E that is Lebesgue measurable but not Borel. Then the function χ_E is Lebesgue measurable but not Borel measurable.

7. If it were Borel then it would be Lebesgue measurable. Which it is not.