Math 4121
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Figure: This is your instructor.

The Lebesgue Integral

Interior Approximation by Closed Sets

Now we consider the idea of approximating a Lebesgue measurable set from the inside by closed sets.

Theorem: *A set $S \subseteq \mathbb{R}$ is Lebesgue measurable if and only if, for each $\epsilon > 0$, there is a closed set $F \subseteq S$ with $m^*(S \setminus F) < \epsilon$.*

Proof: If S is measurable, then its complement ${}^cS$ is also measurable. By the proposition above, there is an open set Ω with ${}^cS \subseteq \Omega$ and $m(\Omega \setminus {}^cS) < \epsilon$. Now we let $F = {}^c\Omega$. Thus F is closed. Also $F \subseteq S$. Further note that $S \setminus F = S \cap \Omega = \Omega \setminus {}^cS$. Hence

$$m(S \setminus F) = m(\Omega \setminus {}^cS) < \epsilon\,.$$

For the converse direction, note that for each $j \in \mathbb{N}$ there is a closed set $F_j \subseteq S$ with $m^*(S \setminus F_j) < 1/j$. Let $K = \cup_{j=1}^{\infty} F_j$. Then K is an F_σ set. So it is Lebesgue measurable. Since $F_j \subseteq K$, we see that $S \setminus K \subseteq S \setminus F_j$. Hence

$$m^*(S \setminus K) \leq m^*(S \setminus F_j) < \frac{1}{j}$$

for all j. Thus $m^*(S \setminus K) = 0$. This implies that $W = S \setminus K$ is a measurable set. In conclusion, $S = K \cup W$ is measurable. $\square$

Corollary: *Let $S \subseteq \mathbb{R}$ be Lebesgue measurable. Then, for any $\epsilon > 0$, there is a closed set $F \subseteq S$ with $m(S) \leq m(F) + \epsilon$. Thus we have*

$$m(S) = \sup\{m(F) : F \text{ closed and } F \subseteq S\}.$$

Proof: The set F in the theorem is measurable and $F \subseteq S$. Thus

$$m(S) = m^*(S \cap F) + m^*(S \setminus F) \leq m(F) + \epsilon\,. \qquad \square$$

Next we characterize measurability in terms of approximation by F_σ sets from the inside.

Corollary: *The following statements are equivalent for a set $E \subseteq \mathbb{R}$.*

(a) *The set E is Lebesgue measurable.*

(b) *There exists an F_σ set K with $K \subseteq E$ and $m^*(E \setminus K) = 0$.*

(c) *There is an F_σ set K and a Lebesgue null set W such that $K \subseteq E$, $W \subseteq E$, and $E = K \cup W$.*

Proof: The proof is a straightforward application of the preceding ideas. We leave the details to the reader. □

Of course a compact set, being bounded, always has finite Lebesgue measure. So it is of interest that we can approximate *virtually any* Lebesgue measurable set from within by compact sets.



Figure: Approximation from the inside by a compact set.

Theorem: *A set $E \subseteq \mathbb{R}$ with $m^*(E) < \infty$ is Lebesgue measurable if and only if, for each $\epsilon > 0$, there is a compact set K such that $K \subseteq E$ and $m^*(E \setminus K) < \epsilon$. See the figure.*

Proof: If E is measurable and $j \in \mathbb{N}$, then we let $E_j = E \cap \{x : |x| \leq j\}$. Since the sequence of sets E_j increases to E, we may conclude that the numerical sequence $\{m(E_j)\}$ increases to $m(E)$. Hence there is an index j_0 so that $m(E) < m(E_{j_0}) + \epsilon/2$. By the theorem above there is a closed set F with $F \subseteq E_{j_0}$ and $m(E_{j_0} \setminus F) < \epsilon/2$.

Of course E is the disjoint union of the sets $E \setminus E_{j_0}$ and E_{j_0} so we have

$$m(E) = m(E \setminus E_{j_0}) + m(E_{j_0}) .$$

Since $m(E) < \infty$, we also know that

$$m(E \setminus E_{j_0}) = m(E) - m(E_{j_0}) < \frac{\epsilon}{2} .$$

Furthermore, $E \setminus F$ is the union of the disjoint sets $E \setminus E_{j_0}$ and $E_{j_0} \setminus F$. So we have

$$m(E \setminus F) = m(E \setminus E_{j_0}) + m(E_{j_0} \setminus F) < \epsilon .$$

Since $F \subseteq E_{j_0}$ is closed and bounded, it is compact. That proves the forward direction of the theorem.

For the converse, suppose that for each $j \in \mathbb{N}$ there is a compact set F_j with $F_j \subseteq E$ and $m^*(E \setminus F_j) < 1/j$. We set $F = \cup_{j=1}^{\infty} F_j$. Then F is measurable and $E \setminus F \subseteq E \setminus F_j$ for each $j \in \mathbb{N}$. Thus we have $m^*(E \setminus F) = 0$. So $W = E \setminus F$ is a Lebesgue null set and so is measurable. In conclusion, $E = F \cup W$ is Lebesgue measurable. $\square$

We conclude this discussion by showing that any Lebesgue measurable set can be approximated by the finite union of bounded, open intervals. In this discussion we use the notation

$$S \triangle T \equiv (S \setminus T) \cup (T \setminus S).$$

Theorem: *Let $E \subseteq \mathbb{R}$ have finite Lebesgue measure. Let $\epsilon > 0$. Then there are bounded open intervals $I_1, I_2, \ldots, I_k$ so that, if $U = \cup_{j=1}^{k} I_j$, then $m(E \triangle U) < \epsilon$.*

Proof: Let $\epsilon > 0$. As we have seen before, there is a collection $\{I_j\}_{j=1}^{\infty}$ of bounded, open intervals which cover E and so that, if $\mathcal{I} = \cup_{j=1}^{\infty} I_j$, then $m(\mathcal{I}) \leq m(E) + \epsilon/2$. We also know that there is a compact set $K \subseteq E$ such that $m(E \setminus K) < \epsilon/2$.

The Heine-Borel theorem tells us that finitely many of the I_j, say $I_1, I_2, \ldots, I_m$, cover K. If we set $L = \cup_{j=1}^{m} I_j$, then $K \subseteq L \subseteq \mathcal{I}$ and $K \subseteq E \subseteq \mathcal{I}$. Therefore

$$\begin{aligned} m(E \triangle L) &= m(E \setminus L) + m(L \setminus E) \\ &\leq m(E \setminus K) + m(\mathcal{I} \setminus E) \\ &< \epsilon \,. \end{aligned}$$

That completes the proof. □

You can check for yourself that this last theorem is also true for closed intervals, or half-open intervals, or even for pairwise disjoint intervals.