Math 4121
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Figure: This is your instructor.

The Lebesgue Integral

Approximation by Open and Closed Sets

The purpose of this discussion is to show that Lebesgue measurable sets are well approximated by open sets and by closed sets. Of course we have to specify the sense or the topology in which these approximations take place.

Lemma: *Let $A \subseteq \mathbb{R}$. Let $\epsilon > 0$. Then there is an open set $U \subseteq \mathbb{R}$ such that $A \subseteq U$ and $m(U) \leq m^*(A) + \epsilon$. See the figure. Thus*

$$m^*(A) = \inf\{m(U) : A \subseteq U, U \text{ open}\}.$$



Figure: Approximation from the outside by an open set.

Lemma: *Let $A \subseteq \mathbb{R}$. Then there is a G_δ set E such that $A \subseteq E$ and $m^*(A) = m(E)$.*

Proof of the First Lemma: We may assume that $m^*(A) < +\infty$. Of course we may require the intervals in the definition of outer measure to be open. So there exists a sequence $\{I_j\}$ of open intervals covering the set A such that

$$\sum_{j=1}^{\infty} \ell(I_j) \leq m^*(A) + \epsilon\,.$$

If we let $\mathcal{O} = \cup_{j=1}^{\infty} I_j$, then $\mathcal{O}$ is open and, by the countable subadditivity of m^* and the fact that $m^* = \ell$ on intervals, we see that

$$m(\mathcal{O}) \leq \sum_{j=1}^{\infty} m^*(I_j) = \sum_{j=1}^{\infty} \ell(I_j) \leq m^*(A) + \epsilon\,.$$

The result now follows from the definition of infimum. □

Proof of the Second Lemma: For each $j \in \mathbb{N}$, let $\mathcal{O}_j$ be an open set such that $A \subseteq \mathcal{O}_j$ and $m(\mathcal{O}_j) \leq m^*(A) + 1/j$. Define $E = \cap_{j=1}^{\infty} \mathcal{O}_j$. Then E is a G_δ.

Thus $A \subseteq E \subseteq \mathcal{O}_j$ for every j. Also $m^*(A) \leq m(E) \leq m^*(A) + 1/j$ for all $j \in \mathbb{N}$. In conclusion, $m^*(A) = m(E)$. □

Corollary: *Any Lebesgue null set is a subset of a Borel null set.*

Proof: Let F be a Lebesgue null set. Then there is a G_δ set H such that $F \subseteq H$ and $m(H) = 0$. But then certainly H is a Borel set. $\square$

It should be noted that, in the second lemma, the set-theoretic difference $E \setminus A$ may not be a small set. We will see later that A is Lebesgue measurable if and only if $E \setminus A$ is a null set.

Proposition: *A set $E \subseteq \mathbb{R}$ is Lebesgue measurable if and only if, for every $\epsilon > 0$, there is an open set $\mathcal{O}$ such that $E \subseteq \mathcal{O}$ and $m^*(\mathcal{O} \setminus E) < \epsilon$.*

Proof: First we assume that E is measurable and $m(E) < \infty$. By the first lemma, there is an open set Ω so that $E \subseteq \Omega$ and $m(\Omega) \leq m(E) + \epsilon$. Since E is measurable and $E \subseteq \Omega$, we see that

$$m(\Omega) = m(\Omega \cap E) + m(\Omega \setminus E) = m(E) + m(\Omega \setminus E)\,.$$

Because $m(E) < \infty$, we have

$$m^*(\mathcal{O} \setminus E) = m(\Omega \setminus E) = m(\Omega) - m(E) < \epsilon\,.$$

In the case that $m(E) = +\infty$, we must work a bit harder. Let $E_1 = E \cap \{x : |x| \leq 1\}$. If $j \geq 2$, let $E_j = E \cap \{x : j - 1 < |x| \leq j\}$. For $j \in \mathbb{N}$, we let Ω_j be an open set with $E_j \subseteq \Omega_j$ and $m(\Omega_j \setminus E_j) < \epsilon/2^j$. Set $\Omega = \cup_{j=1}^{\infty} \Omega_j$. Then Ω is open, $E \subseteq \Omega$, and

$$\Omega \setminus E \subseteq \bigcup_{j=1}^{\infty} (\Omega_j \setminus E_j)\,.$$

As a result, the countable subadditivity of m tells us that

$$m^*(\mathcal{O} \setminus E) = m(\Omega \setminus E) \leq \sum_{j=1}^{\infty} m(\Omega_j \setminus E_j) < \sum_{j=1}^{\infty} \frac{\epsilon}{2^j} = \epsilon \, .$$

Conversely, assume that for every $j \in \mathbb{N}$ there is an open set $\Omega_j \supseteq E$ so that $m^*(\Omega_j \setminus E) < 1/j$. Define $U = \cap_{j=1}^{\infty} \Omega_j$. Thus U is a G_δ set. So it is measurable. Also, since $U \subseteq \Omega_j$ for each j, we see that $U \setminus E \subseteq \Omega_j \setminus E$. Thus

$$0 \leq m^*(U \setminus E) \leq m^*(\Omega_j \setminus E) < \frac{1}{j}$$

for all $j \in \mathbb{N}$. Thus $m^*(U \setminus E) = 0$. And therefore $W = U \setminus E$ is a measurable set. And so $E = U \setminus W$ is a measurable set. $\square$

Corollary: *Let $E \subseteq \mathbb{R}$ be a measurable set. For any $\epsilon > 0$ there then exists an open set $\Omega \supseteq E$ with $m(\Omega) \leq m(E) + \epsilon$. Thus*

$$m(E) = \inf\{m(\Omega) : \Omega \text{ is open and } \Omega \supseteq E\}.$$

Proof: The proposition tells us that

$$m(\Omega) = m(E) + m(\Omega \setminus E) \leq m(E) + \epsilon. \qquad \square$$

We next consider the approximation of a Lebesgue measurable set from the outside by a G_δ. This turns out to be a useful characterization of Lebesgue measurability.

Proposition: *The following statements are logically equivalent for a set $E \subseteq \mathbb{R}$.*

(a) *The set $E \subseteq \mathbb{R}$ is Lebesgue measurable.*

(b) *There exists a G_δ set W with $E \subseteq W$ and $m^*(W \setminus E) = 0$.*

(c) *There exists a G_δ set W and a Lebesgue null set S so that $E \subseteq W$, $S \subseteq W$, and $E = W \setminus S$.*

Proof: This proof is a straightforward application of the preceding ideas. We leave the details to the reader. □