Math 4121
January 27, 2021 Lecture

Steven G. Krantz

January 13, 2021



Figure: This is your instructor.

The Lebesgue Integral

The Idea of the Lebesgue Integral

Let $\widetilde{C}$ be a set constructed like the Cantor ternary set (see the review below) in which intervals of length 5^{-j} are removed at each step. Then $[0,1] \setminus \widetilde{C}$ has length less than 1 and hence $\widetilde{C}$ has positive measure. Consider $f(x) = \chi_{\widetilde{C}}(x)$. Then, by our theorem, f is *not* Riemann integrable. But it *is* Lebesgue integrable.

The Idea of the Lebesgue Integral

As you saw in the last lecture, the Riemann integral is predicated on the idea of breaking up the *domain* of the function f. By contrast, the Lebesgue integral (as we shall see below) is predicated on the idea of breaking up the *range* of the function f.

First we need to discuss the idea of measure. Our dream is to be able to measure the length of any set of reals. Certainly we know how to measure the length of an interval $[a, b]$. It is of course $b - a$. And we can measure the length of a finite union of intervals

$$S = [a_1, b_1] \cup [a_2, b_2] \cup \cdots \cup [a_k, b_k]$$

with

$$a_1 < b_1 < a_2 < b_2 < \cdots < a_k < b_k \, .$$

The answer, of course, is

$$\sum_{j=1}^{k} (b_j - a_j) \, .$$

See Figure 1.3.



Figure: The length of a set.

But what about more complicated sets? For example, what is the length of the Cantor ternary set? Refer to Figures 1.4, 1.5. 1.6 to remind yourself how the Cantor ternary set is constructed. Recall that we do so by intersecting

$$\begin{aligned} I_0 &= [0,1] \\ I_1 &= [0,1/3] \cup [2/3,1] \\ I_2 &= [0,1/9] \cup [2/9,1/3] \cup [2/3,7/9] \cup [8/9,1] \end{aligned}$$

etc. The Cantor set is uncountable, has length 0, and is perfect.



Figure: First step in the construction of the Cantor set.



Figure: Second step in the construction of the Cantor set.



Figure: Third step in the construction of the Cantor set.

An idea that goes back to the mid-nineteenth century is this. Suppose that $E \subset \mathbb{R}$ is a closed, bounded set. Then E is contained in a large open interval I. And $I \setminus E$ is an open set, so it is a union of open intervals. We can measure the length of each of those open intervals, and then subtract all those lengths from the length of I. And that will give us the length of E.

The reasoning in the last paragraph is correct, but it fails to treat a great many sets which we would like to measure. We need a technique for calculating length which treats all sets at once. But it turns out that there is a significant roadblock to this program. In fact *it is impossible to assign a length to every set of real numbers.*

How can this be? In order to answer this question, we should first consider what properties we would want a notion of length to possess. Let $m(S)$ denote the purported length of the set S. We would want m to have these properties:

- $m(S) \geq 0$ for every set S;
- $m(A \cup B) \leq m(A) + m(B)$ (subadditivity—see Figure 1.7);
- If $A_1, A_2, \ldots$ are countably many pairwise disjoint sets then

$$m\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} m(A_j)\,.$$

- If $a \in \mathbb{R}$ then define $\tau_a(S) = \{s + a : s \in S\}$ to be the translation of S by a units. Then we mandate that $m(\tau_a(S)) = m(S)$. In other words, the measure should be *translation invariant*.

We note in passing that the third property here is called *countable additivity*.

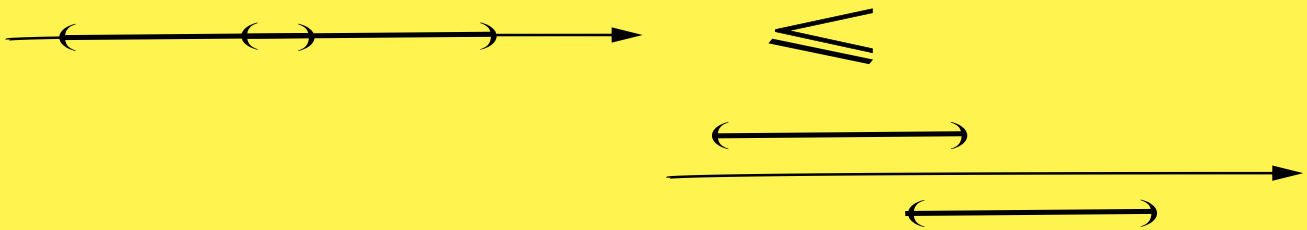


Figure: Subadditivity.

Example: We work in the real numbers $\mathbb{R}$. Let $A = [0, 3/4]$ and $B = [1/4, 1]$. Then $A \cup B = [0, 1]$. So

$$\begin{aligned} 1 &= m(A \cup B) \\ &= m([0,1]) \\ &\leq m([0,3/4]) + m([1/4,1]) \\ &= m(A) + m(B) \\ &= \frac{3}{4} + \frac{3}{4} \\ &= \frac{3}{2}. \end{aligned}$$

We in fact have strict inequality because A and B have nontrivial overlap.

Remark: It will be common in this book for us to consider a sum of the form

$$\sum_{j=1}^{\infty} a_j \,,$$

where each $a_j = 0$. The value of such an infinite sum, or series, must be 0. For that value is *defined* to be the limit as $N \to +\infty$ of the partial sums

$$S_N = \sum_{j=1}^{N} a_j \,,$$

and each of those partial sums is equal to 0.

It turns out that the following is the case:

Theorem: There is no notion of length m that will satisfy all three of the above bulleted properties for *every* set of real numbers.

Proof: We perform this construction on the unit interval $I = [0, 1]$. We do arithmetic on I modulo 1. That is, when we add two numbers together, we subtract off any multiples of 1 to get the answer. Thus

$$\frac{3}{4} + \frac{3}{4} = \frac{1}{2} \quad , \quad \frac{1}{2} + \frac{5}{8} = \frac{1}{8}$$

are two examples.

Now define a relation on I by $x \sim y$ if $x - y$ is rational. It is easy to check that this is an equivalence relation. There are uncountably many equivalence classes, and each equivalence class has countably many elements. We use the Axiom of Choice to select one element from each equivalence class. Let S be the set of those selected elements. We claim that S is the nonmeasurable set that we seek.

If q is a nonzero rational number then consider the set $S + q = \{s + q : s \in S\}$. We claim that $S + q$ is disjoint from $S + q'$ for q, q' distinct rational numbers. In fact if $x \in S + q \cap S + q'$, then $x = s + q$ and $x = s' + q'$ for some $s, s' \in S$. But this would mean that $s - s'$ is rational. Hence s and s' are in the same equivalence class. But they are not! The only way that $s - s'$ could be rational is if $s = s'$. But then $q = q'$. Hence our two sets $S + q$ an $S' + q'$ are actually the same.

So $S + q$ and $S + q'$ are disjoint. And these two sets are geometrically identical—one is just a translate of the other.

We know that

$$I = \bigcup_{q \in [0,1] \cap \mathbb{Q}} S + q\,. \tag{1}$$

This is just because every element of I has the form $s + q$ for some $s \in S$ and some $q \in \mathbb{Q}$.

Now what is the measure of S? If it is 0, then each $S + q$ also has measure 0 and then it follows from the displayed equation that I has measure 0. That is of course false. If instead S has positive measure σ then each $S + q$ has positive measure σ. But then it follows from our equation that I has infinite measure. That, too, is false. So there is no measure that we can logically assign to S. □

It is interesting that we used the Axiom of Choice to construct the set S above. In fact Solovay has shown that, if you deny the Axiom of Choice, then it is possible to assign a measure to *every* set of reals.

It turns out that we will be able to identify an algebra of sets which we call *measurable*. It is the measurable sets that we can actually measure; other sets we do not attempt to measure. Fortunately, the collection of measurable sets is quite robust and is closed under reasonable mathematical operations. It is a fairly straightforward collection of objects. We shall begin to develop these ideas in the next section.