Math 4121
January 25, 2021 Lecture

Steven G. Krantz

January 25, 2021



Figure: This is your instructor.

The Lebesgue Integral

Review of the Riemann Integral

In this course we focus our attention on real-valued functions. We give an occasional nod to complex-valued functions.

Going back to your days of learning calculus, you have been and are now familiar with the Riemann integral. This is the integral that is modeled on *Riemann sums*. Just to review:

Let $[a, b] \subseteq \mathbb{R}$ be a bounded, closed interval and let $f : [a, b] \to \mathbb{R}$ be a continuous function. A *partition* of $[a, b]$ is a sequence of points $\mathcal{P} = \{x_0, x_1, \ldots, x_k\}$ with

$$a = x_0 \leq x_1 \leq x_2 \leq \cdots \leq x_k = b\,.$$

We let $I_j = [x_{j-1}, x_j]$ be the jth interval in the partition, $j = 1, 2, \ldots, k$. Let $\triangle_j = |x_j - x_{j-1}|$ be the length of the jth interval. Define the *mesh* of the partition to be

$$m(\mathcal{P}) = \max_{j=1,\ldots,k} \triangle_j\,.$$

For each j, select a point $\xi_j \in I_j$. Refer to Figure 1.1.

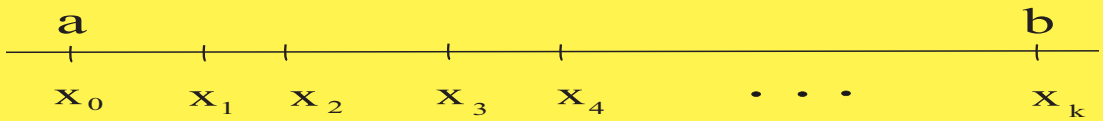
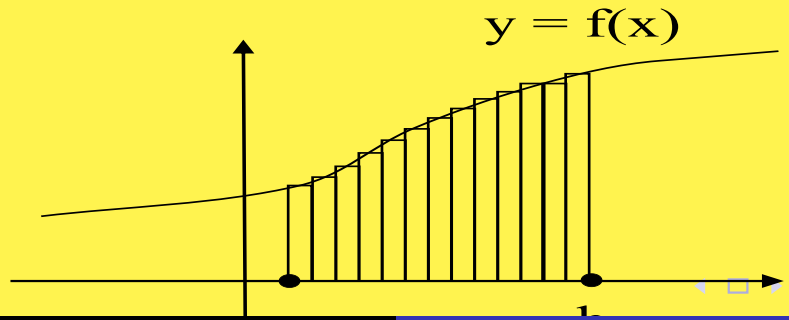


Figure: A partition.

We define the *Riemann sum* of the function f based on the partition $\mathcal{P}$ to be

$$\mathcal{R}_{\mathcal{P}} = \sum_{j=1}^{k} f(\xi_j) \cdot \triangle_j \, .$$

We think of this sum as giving an approximation to the area under the graph of f and between the left-right limits a and b. See Figure 1.2.



If the limit

$$\lim_{m(\mathcal{P})\to 0} \mathcal{R}_{\mathcal{P}}$$

exists, then we call the limit the *Riemann integral of the function f on the interval $[a, b]$* and we denote it by

$$\int_a^b f(x)\,dx\,.$$

Now the most fundamental result about the Riemann integral is this:

Theorem: Let f be a continuous function on the interval $[a, b]$. Then the Riemann integral

$$\int_a^b f(x)\,dx$$

exists.

We shall not provide the proof of this theorem here, but instead refer the reader to the references.

It is not difficult to see that a *piecewise continuous* function will also be integrable. In fact the definitive result about Riemann integrability is the following:

Theorem: A function f on a closed interval $[a, b]$ is Riemann integrable if and only if the set of discontinuities of f has measure 0.

Of course we do not really know yet what "set of measure zero" means; in fact this is one of the big ideas that we are going to learn in this course. But, for the moment, you can take "measure 0" to mean "zero length."

The big feature that differentiates the Lebesgue integral from the Riemann integral is the way that these two theories treat limits. The basic limit theorem for the Riemann integral is this.

Theorem: Let f_j be Riemann integrable functions on a bounded interval $[a, b]$. If $f_j \to f$ uniformly as $j \to \infty$, then

$$\int_a^b f_j(x)\,dx \to \int_a^b f(x)\,dx$$

as $j \to \infty$.

We now present a couple of examples to suggest the shortcomings of the Riemann integral.

Example: In what follows, if $E \subseteq \mathbb{R}$ is a set, then we let the *characteristic function* χ_E of E be defined by

$$\chi_E(x) = \begin{cases} 1 & \text{if} \quad x \in E\,, \\ 0 & \text{if} \quad x \notin E\,. \end{cases}$$

Now define

$$f_j(x) = \chi_{[j,j+1]}(x)\,.$$

These functions *do not* converge uniformly. So the Riemann theory can say nothing about the sequence of integrals $\int f_j(x)\,dx$. It is nevertheless clear that the indicated integrals converge to 1 as $j \to +\infty$. But, as we shall see below, the Lebesgue theory can say something about the sequence of integrals.

In fact we note here, with reference to our earlier theorem, that

$$\begin{aligned} 0 &= \int \lim_{j\to+\infty} f_j(x)\,dx \\ &\neq \lim_{j\to+\infty} \int f_j(x)\,dx \\ &= 1\,. \end{aligned}$$

As a second example, let

$$g_j(x) = \chi_{[1,j]} \, .$$

These functions *do not* converge uniformly. So the Riemann theory can say nothing about the sequence of integrals $\int g_j(x)\, dx$. [It is nevertheless clear that the indicated integrals converge to $+\infty$.] But, as we shall see below, the Lebesgue theory can. Again, with reference to our theorem, we note that

$$\begin{aligned} +\infty &= \int \chi_{[1,+\infty)} \, dx \\ &= \int \lim_{j\to+\infty} g_j(x)\, dx \\ &= \lim_{j\to+\infty} \int g_j(x)\, dx \\ &= \lim_{j\to+\infty} (j-1) = +\infty \, . \end{aligned}$$

Next, let

$$h_j(x) = \sum_{\ell=1}^{j} \chi_{[0,2^{-\ell}]} \, .$$

Intuitively, the integrals $\int h_j(x)\,dx$ converge to $\sum_{\ell=1}^{\infty} 2^{-\ell} = 1$. But the functions do not converge uniformly so the Riemann integral cannot give this result. We note that, for this example, both sides of the display in the theorem equal $\sum_{\ell=1}^{\infty} 2^{-\ell}$.

Now we let

$$m_j(x) = j \cdot \chi_{[j, j+1/j^3]} \,.$$

Although it is intuitively clear that $\int m_j(x)\, dx$ converges as $j \to +\infty$, the Riemann integral will not give this result. With reference to the theorem, we see that

$$0 = \lim_{j \to +\infty} \int m_j(x)\, dx = \int \lim_{j \to +\infty} m_j(x)\, dx = 0 \,.$$

Example: Refer to the last example for notation.

Let $\{q_j\}$ be an enumeration of the rational numbers. Define

$$f_1(x) = \chi_{\{q_1\}},$$

$$f_2(x) = \chi_{\{q_1\}} + \chi_{\{q_2\}},$$

$$f_3(x) = \chi_{\{q_1\}} + \chi_{\{q_2\}} + \chi_{\{q_3\}},$$

etc. Then the sequence of functions $\{f_j\}$ converges to $f(x) = \chi_{\mathbb{Q}}$. Each of the functions f_j is Riemann integrable by the theorem. But the function f is *not* Riemann integrable. It will turn out that the function f *is* Lebesgue measurable and Lebesgue integrable.