# Math 4111

## SOLUTIONS TO PRACTICE MIDTERM 2

1. Let $S = \{x \in \mathbb{R} : 1 < x < 4\}$. Then let $s \in S$. Set $\varepsilon = \min\{s-1, 4-s\}$. Then $(s-\varepsilon, s+\varepsilon) \subseteq S$. So $S$ is open.

2. Let $\mathcal{O} = \bigcup_j O_j$. If $x \in \mathcal{O}$ then $x \in O_j$ for some $j$. Since $O_j$ is open, $\exists\, \varepsilon > 0$ such that $(x-\varepsilon, x+\varepsilon) \subseteq O_j$. But then $(x-\varepsilon, x+\varepsilon) \subseteq \bigcup_j O_j = \mathcal{O}$. So $\mathcal{O}$ is open.

3. Let $E, F$ be compact. Then $E, F$ are closed and bounded. Hence $E \cup F$ is closed and bounded. So $E \cup F$ is compact.

4. Work in the unit interval. Let $I = [0,1]$. Let

$$S_1 = I \setminus \left(\frac{3}{8}, \frac{5}{8}\right)$$

$$S_2 = S_1 \setminus \left[\left(\frac{7}{32}, \frac{9}{32}\right) \cup \left(\frac{23}{32}, \frac{25}{32}\right)\right]$$

etc,

In the first step we remove one interval of length $\frac{1}{4}$.
In the second step we remove two intervals of length $\frac{1}{16}$.
$\vdots$
In the $j^{\text{th}}$ step we remove $2^{j-1}$ intervals of length $\left(\frac{1}{4}\right)^j$

So the sum of the lengths of all removed intervals is

$$\sum_{j=1}^{\infty} 2^{j-1} \cdot \left(\frac{1}{4}\right)^j = \frac{1}{4} \sum_{j=1}^{\infty} \left(\frac{2}{4}\right)^{j-1}$$

$$= \frac{1}{4} \sum_{j=0}^{\infty} \left(\frac{1}{2}\right)^j$$

$$= \frac{1}{4} \cdot \frac{1}{1-1/2} = \frac{1}{4} \cdot 2 = \frac{1}{2}.$$

Hence the remaining Cantor set has length $1/2$.

5. a) The left and right limits at 0 exist but are unequal to $f(0)$. So it's a discontinuity of the first kind.

b) The left limit at 0 does not exist. So it's a discontinuity of the second kind.

c) The left and right limits of $h$ at 0 exist but are unequal to $h(0)$. So it's a discontinuity of the first kind.

6. By l'Hôpital's Rule applied twice, we get

$$\lim_{x \to 0} \frac{\sin x}{2x} = \frac{1}{2} \lim_{x \to 0} \frac{\sin x}{x} = \frac{1}{2} \lim_{x \to 0} \frac{\cos x}{1} = \frac{1}{2}.$$

7. $f(3) - f(0) = (3-0)\cdot f'(\xi)$, some $0 < \xi < 3$.

So

$$|f(3)| \le |f(0)| + 3\cdot 4$$
$$= 2 + 12 = 14.$$

8. Let $\psi$ be as in the book. Define

$$F(x) = \int_0^x \psi(t)\,dt.$$

Let $$f(x) = \sum_{j=1}^{\infty} \left(\frac{3}{16}\right)^j F(4^j x).$$

Then $f'$ exists at each $x$ and equals

$$f'(x) = \sum_{j=1}^{\infty} \left(\frac{3}{4}\right)^j F'(4^j x)$$
$$= \sum_{j=1}^{\infty} \left(\frac{3}{4}\right)^j \psi(4^j x).$$

But this $\uparrow$ is the Weierstrass nowhere differentiable function. So no second derivatives exist.

9. Let $f(x) = \sqrt{x}$. We are looking at

$$f(x+5) - f(x) = [(x+5) - x]\cdot f'(\xi),$$

some $x < \xi < x+5$. Of course $f'(\xi) = \frac{1}{2}\xi^{-1/2} < \frac{1}{2}x^{-1/2}$.

Hence $f(x+5) - f(x) \le 5\cdot \frac{1}{2}x^{-1/2} \to 0$.

20. $f(x) = \begin{cases} x^5 & \text{for } x \geq 0 \\ -x^5 & \text{for } x < 0. \end{cases}$

Thus

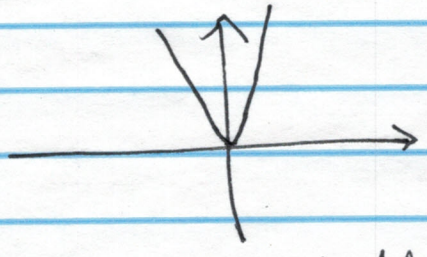
$$f'(x) = \begin{cases} 5x^4 & \text{for } x \geq 0 \\ -5x^4 & \text{for } x < 0 \end{cases}$$

$$f''(x) = \begin{cases} 20x^3 & \text{for } x \geq 0 \\ -20x^3 & \text{for } x < 0 \end{cases}$$

$$f'''(x) = \begin{cases} 60x^2 & \text{for } x \geq 0 \\ -60x^2 & \text{for } x < 0 \end{cases}$$

$$f''''(x) = \begin{cases} 120x & \text{for } x \geq 0 \\ -120x & \text{for } x < 0 \end{cases}$$

The graph of $f''''$ looks like



So it is <u>not</u> differentiable at 0.

Thus $f$ is $C^4$ but not $C^5$. We see that $f''''$ is Lipschitz with Lipschitz constant 120. So $f \in C^{4,1}$.

11. Let $f(x) = x^x$. Set $g(x) = \log f(x) = x \log x$

$= \frac{\log x}{1/x}$. Both numerator and denominator blow up as $x \to 0$. So l'Hôpital applies.

We get

$$\lim_{x\to 0} \frac{\log x}{1/x} = \lim_{x\to 0} \frac{1/x}{-1/x^2} = \lim_{x\to 0} -x = 0.$$

Thus $\lim_{x\to 0} g(x) = 0$ hence $\lim_{x\to 0} f(x) = 1$.

22. Select $e_j \in E$, $k_j \in K$ such that

$$|e_j - k_j| \to \inf\{|e-k| : e\in E, k\in K\}.$$

Since $\{k_j\} \subseteq K$ compact, there is a convergent subsequence $k_{j_l}$. Say $k_{j_l} \to k_0$.

But then $\{e_{j_l}\}$ is bounded since

$$|e_{j_l}| \le |e_{j_l} - k_{j_l}| + |k_{j_l} - k_0| + |k_0|$$

which is less than $\left(\inf_{\substack{e\in E\\ k\in K}} |e-k| + \frac{\varepsilon}{2}\right) + \frac{\varepsilon}{2} + |k_0|$

if $l$ is large enough.

So $e_{j_l}$ has convergent subsequence $e_{j_{l_m}}$ with limit $e_0$. And $e_0$ must lie in $E$ since $E$ is closed. But $E \cap K = \emptyset$, so $e_0 \neq k_0$.

And $|e_0 - k_0| = \inf\{|e-k| : e\in E, k\in K\}$.

Let $\varepsilon = \frac{1}{2}|e_0 - k_0|$. That gives the result.