Math 4111 Fall, 2020
Krantz

PRACTICE EXAM FOR SECOND MIDTERM

1. Give a rigorous proof that the set $\{x \in \mathbb{R} : 1 < x < 4\}$ is open.

2. Prove that if O_j are open sets, $j = 1, 2, 3, \ldots$, then $\cup_j O_j$ is open.

3. Prove that the union of two compact sets is compact.

4. Construct a Cantor-like set that has length $1/2$.

5. In each of the following, explain whether the discontinuity at the origin is of the first kind or the second kind. Give an explicit reason for your answer in each case.

(a)

$$f(x) = \begin{cases} 3x & \text{if} \quad x < 0 \\ 1 & \text{if} \quad x = 0 \\ 4x & \text{if} \quad x > 0\,. \end{cases}$$

(b)

$$g(x) = \begin{cases} \cos(1/x) & \text{if} \quad x < 0 \\ 1 & \text{if} \quad x = 0 \\ x\sin(1/x) & \text{if} \quad x > 0\,. \end{cases}$$

(c)

$$h(x) = \begin{cases} 4 & \text{if} \quad x < 0 \\ 5 & \text{if} \quad x = 0 \\ 3x & \text{if} \quad x > 0\,. \end{cases}$$

6. Give a rigorous proof that

$$\lim_{x \to 0} \frac{1 - \cos x}{x^2} = \frac{1}{2}\,.$$

7. A function f with domain the entire real line is differentiable. It satisfies $|f'(x)| \leq 4$ for all x. It is known that $f(0) = 2$. Give an estimate for the size of $f(3)$.

8. Modify the Weierstrass nowhere differentiable function to obtain a function that is differentiable at every point but has no second derivative at any point.

9. Prove that $\lim_{x\to+\infty}(\sqrt{x+5}-\sqrt{x})=0$.

10. Let $f(x)=|x|^5$. In which class C^k does it lie? In which class $C^{k,\alpha}$ does it lie?

11. Calculate $\lim_{x\to 0} x^x$.

12. Let K be compact and E closed. Assume that $K\cap E=\emptyset$. Prove that there exists an $\epsilon>0$ such that, for any $k\in K$ and $e\in E$, $|k-e|\geq\epsilon$.