Math 4111 Fall, 2020
Krantz

SECOND MIDTERM EXAM

General Instructions: Read the statement of each problem carefully. If you want full credit on a problem then you must show your work. If you only write the answer then you will *not* receive full credit.

Be sure to ask questions if anything is unclear.

(8 points) **1.** Give a rigorous proof that the set $\{x \in \mathbb{R} : 1 < x^2 < 2\}$ is open.

(8 points) **2.** Prove that if E_j are closed sets, $j = 1, 2, 3, \ldots$, then $\cap_j E_j$ is closed.

(8 points) **3.** Prove that the intersection of two compact sets is compact.

(8 points) **4.** What is the interior of the Cantor set? What is the boundary?

(9 points) **5.** In each of the following, explain whether the discontinuity at the origin is of the first kind or the second kind. Give an explicit reason for your answer in each case.

(a)

$$f(x) = \begin{cases} x^2 & \text{if} \quad x < 0 \\ 1 & \text{if} \quad x = 0 \\ 2x & \text{if} \quad x > 0\,. \end{cases}$$

(b)

$$g(x) = \begin{cases} x \sin(1/x) & \text{if} \quad x < 0 \\ 0 & \text{if} \quad x = 0 \\ \sin(1/x) & \text{if} \quad x > 0\,. \end{cases}$$

(c)

$$h(x) = \begin{cases} 1 & \text{if} \quad x < 0 \\ 2 & \text{if} \quad x = 0 \\ 3 & \text{if} \quad x > 0\,. \end{cases}$$

(8 points) **6.** Give a rigorous proof that

$$\lim_{x \to 0} \frac{\sin x}{x} = 1\,.$$

(8 points) **7.** A function f with domain the entire real line is differentiable. It satisfies $|f'(x)| \leq 1$ for all x. It is known that $f(0) = 0$. Give an estimate for the size of $f(3)$.

(9 points) **8.** A function f is differentiable on the entire interval $(-1, 1)$. It is known that f' satisfies the intermediate value property, even though f' need not be continuous. Explain why this is so. Give a relevant example.

(9 points) **9.** Prove that $\lim_{x \to +\infty} (\sqrt{x+1} - \sqrt{x}) = 0$.

(9 points) **10.** Let $f(x) = |x|^3$. In which class C^k does it lie? In which class $C^{k,\alpha}$ does it lie?

(8 points) **11.** Calculate $\lim_{x \to +\infty} x^{1/x}$.

(8 points) **12.** Let K be a compact set. Define

$$f(x) = \inf\{|x - k| : k \in K\}\,.$$

Prove that f is a Lipschitz function.