Math 4111 Fall, 2020
Krantz

FIRST MIDTERM EXAM

General Instructions: Read the statement of each problem carefully. If you want full credit on a problem then you must show your work. If you only write the answer then you will *not* receive full credit.

Be sure to ask questions if anything is unclear. This exam has 8 questions and is worth 100 points. You will have 50 minutes to take this exam.

(8 points) **1.** Prove that the sequence

$$a_j = \frac{(-1)^j \cdot 2}{j+1}$$

converges to 0.

Let $\varepsilon > 0$. Let $N > \frac{2}{\varepsilon}$. If $j > N$ then

$$|a_j - 0| = \left|\frac{(-1)^j \cdot 2}{j+1} - 0\right| = \frac{2}{j+1} < \frac{2}{j} < \frac{2}{N} < \varepsilon.$$

(8 points) **2.** What is the multiplicative inverse of the complex number $2 + 3i$?

It is $\frac{2-3i}{2^2+3^2} = \frac{2}{13} - \frac{3}{13}i$

(8 points) **3.** Find all cube roots of the complex number $-1 + i$.

$-1+i = \sqrt{2}\, e^{i\frac{3\pi}{4}}$.

$(re^{i\theta})^3 = \sqrt{2}\, e^{i\frac{3\pi}{4}} \Rightarrow r = 2^{1/6}, \theta = \frac{\pi}{4}, z_1 = 2^{1/6} e^{i\frac{\pi}{4}}$.

$(re^{i\theta})^3 = \sqrt{2}\, e^{i(\frac{3\pi}{4}+2\pi)} \Rightarrow r = 2^{1/6}, \theta = \frac{11\pi}{12}, z_2 = 2^{1/6} e^{i\frac{11\pi}{12}}$

$(re^{i\theta})^3 = \sqrt{2}\, e^{i(\frac{3\pi}{4}+4\pi)} \Rightarrow r = 2^{1/6}, \theta = \frac{19\pi}{12}, z_3 = 2^{1/6} e^{i\frac{19\pi}{12}}$

(8 points) **4.** What is the least upper bound of the set $\{x \in \mathbb{R} : x^2 < 7\}$? Give a reason for your answer.

The least upper bound is $\sqrt{7} \in \mathbb{R}$. If $x < \sqrt{7}$. Then $x^2 < 7$ so not an upper bound. If $x > \sqrt{7}$ then $x^2 > 7$ so not the least upper bound.

(8 points) **5.** Prove that if

$$\liminf_{j \to +\infty} a_j = \limsup_{j \to +\infty} a_j ,$$

then the sequence $\{a_j\}$ converges.

If a_{j_k} is any subsequence then

$$\liminf_{j \to \infty} a_j \leq \liminf_{k \to \infty} a_{j_k} \leq \limsup_{k \to \infty} a_{j_k} \leq \limsup_{j \to \infty} a_j .$$

But left and right are equal so they are all equal. Every subsequence converges to the same limit. So the entire sequence does so as well.

(9 points) **6.** Discuss convergence and divergence for each of these series.

(a) $\sum_{j=1}^{\infty} \frac{j!}{j^j}$

(b) $\sum_{j=1}^{\infty} \frac{(2j)!}{(3j)!}$

(c) $\sum_{j=2}^{\infty} \frac{1}{j \cdot \log^2 j}$

(a) $\frac{j!}{j^j} = \frac{j(j-1)(j-2)\cdots 1}{j \cdot j \cdot j \cdots j} < \frac{\overbrace{j/2 \cdot j/2 \cdots j/2}^{j/2 \text{ times}}}{\underbrace{j \cdot j \cdots j}_{j/2 \text{ times}}} = \left(\frac{1}{2}\right)^{j/2}$

and $\sum \left(\frac{1}{2}\right)^{j/2}$ converges

(b) $\frac{(2j)!}{(3j)!} < \frac{(2j)(2j-1)\cdots j}{3j\,(3j-1)\cdots 2j} < \left(\frac{2}{3}\right)^j$ and $\sum \left(\frac{2}{3}\right)^j$ converges.

(c) converges by the integral test.

(9 points) **7.** Let $\sum_j a_j$ and $\sum_j b_j$ be series of positive terms. Prove that, if there is a constant $C > 0$ such that

$$\frac{1}{C} \leq \frac{a_j}{b_j} \leq C$$

for all j large, then either both series converge or both series diverge.

We see that $a_j \leq C b_j$. So if $\sum b_j$ converges then $\sum a_j$ converges by comparison test.

Likewise $b_j \leq C a_j$. So if $\sum a_j$ converges then $\sum b_j$ converges.

(9 points) **8.** Discuss convergence or divergence of the series

$$\sum_{j=1}^{\infty} \frac{\cos j}{j}.$$

$$\sum_{j=1}^{n} \cos j = \mathrm{Re} \sum_{j=1}^{n} e^{ij} = \mathrm{Re} \frac{1 - e^{i(n+1)}}{1 - e^{i}}.$$

So apply Abel's convergence Test just like we did for $\sum \frac{\sin j}{j}$.

(8 points) **9.** Let $\sum_j a_j$ and $\sum_j b_j$ be convergent series of positive real numbers. Discuss the convergence of $\sum_j (a_j \cdot b_j)$.

Since $\sum b_j$ converges, $b_j \to 0$. So, for j large, $0 < b_j < \frac{1}{2}$. Then $0 < a_j b_j < \frac{a_j}{2}$.

So, by comparison test, $\sum a_j b_j$ converges.

(9 points) **10.** If $1/2 > b_j > 0$ and if $\sum_j b_j$ converges then prove that

$$\sum_{j=1}^{\infty} \frac{b_j}{1-b_j}$$

converges.

We see that $0 < \frac{b_j}{1-b_j} < 2b_j$.

So $\sum \frac{b_j}{1-b_j}$ converges by the Comparison test.

(8 points) **11.** Let $\gamma > 0$ be a fixed real number. Give an example of a sequence whose limsup and liminf differ by γ.

Let $a_j = \frac{\gamma}{2}(-1)^j$. Assume $\gamma > 0$.

Then $\limsup a_j = \frac{\gamma}{2}$

$\liminf a_j = -\frac{\gamma}{2}$.

Proof is similar if $\gamma < 0$.

(8 points) **12.** Consider $\{a_j\}$ both as a sequence and as a set. How are the limsup of the sequence and the sup of the set related? How are the liminf of the sequence and the inf of the set related?

Let $a_j = \frac{1}{j}$, $j = 1, 2, \ldots$. Then

$\sup\{a_j\} = 1$, $\limsup a_j = 0$.

Let $a_j = -\frac{1}{j}$, $j = 1, 2, \ldots$ Then

$\inf\{a_j\} = -1$

$\liminf a_j = 0$.