MATH 4111

SOLUTIONS TO HW6

§6.3 1. $\left|\dfrac{f(x+h)+f(x-h)-2f(x)}{h^2}\right|$

$= \left|\dfrac{[f(x+h)-f(x)]-[f(x)-f(x-h)]}{h^2}\right|$

$= \left|\dfrac{h\cdot f'(\xi_1)-h\cdot f'(\xi_2)}{h^2}\right|$ by MVT

$= \dfrac{|f'(\xi_1)-f'(\xi_2)|}{|h|}$

$= \dfrac{|f''(\xi_3)(\xi_1-\xi_2)|}{|h|\cdot}$ (*) by MVT

But ξ_1 is between x & $x+h$

ξ_2 is between x & $x-h$

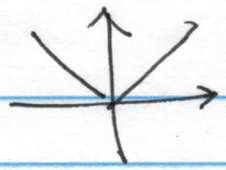
So $|\xi_1-\xi_2| \le 2|h|$. Thus

$(*) \le \dfrac{2|h||f''(\xi_3)|}{|h\cdot 1} \le 2C.$

2. Let $f(x) = \begin{cases} -1 & \text{if } x \le 0 \\ 1 & \text{if } x > 0 \end{cases}$

$g(x) = \begin{cases} -1 & \text{if } x \le 0 \\ 1 & \text{if } x > 0. \end{cases}$

Then f, g not in any C^k class, but $f\cdot g \equiv 1$ is C^∞.

3. If $\ell = 1$ then f is 
This is in $C^{0,1}$.

For $\ell > 1$, by induction, f is in $C^{\ell-1,1}$.

8. Let $g(x) = x^{1/x}$. Define $f(x) = \log g(x) = \frac{1}{x}\log x$
$= \frac{\log x}{x}$.

Then numerator & denominator both tend to $+\infty$, so l'Hôpital applies

$$\lim_{x\to+\infty} f(x) = \frac{1/x}{1} = 0.$$

So $\lim_{x\to+\infty} g(x) = 1$.

§7.1 # 1, Suppose to the contrary that f on $[a,b]$ is unbounded. Say that $[a,b] \to p_j \to x_0 \in [a,b]$ and $f(p_j) \to \infty$. Consider the partition

$P = \{a, x_0-\varepsilon, x_0+\varepsilon, b\}$.

Then $R(f,P) = [(x_0-\varepsilon)-a]\cdot f(a) + 2\varepsilon f(p_j) + [b-(x_0+\varepsilon)]f(b$

and this can be arbitrarily large.

2. Let $P = \{a_0, a_1, \dots, a_k\}$ be a partition of $[a, b]$.

Then
$$R(f, P) = \sum_{j=1}^{k} f(s_j)\,\Delta_j$$

can be 0 or 1, no matter how small the mesh of the partition. So

$$\lim_{\text{mesh}(P) \to 0} R(f, P) \quad \text{does not exist.}$$

6. $P = \{1, \frac{3}{2}, 2, \frac{5}{2}, 3\}$.

$$R(f, P) = \frac{1}{2}\sin(\pi 1^2) + \frac{1}{2}\sin\left(\pi\left(\tfrac{3}{2}\right)^2\right) + \frac{1}{2}\sin(\pi \cdot 2^2) + \frac{1}{2}\sin\left(\pi\left(\tfrac{5}{2}\right)^2\right)$$

7. Let the discontinuities be at $p_1, p_2, \dots, p_m$. Say that

$$a < p_1 < p_2 < \cdots < p_m < b.$$

Since $\lim_{x \to p_j^-} f(x)$ and $\lim_{x \to p_j^+} f(x)$ both exist for each p_j, we may think of f as continuous on each interval.

$$[a, p_1], [p_1, p_2], [p_2, p_3], \dots, [p_{m-1}, p_m], [p_m, b].$$

So, by Theorem 7.10, f is integrable on each of these intervals. Hence

$$\int_a^b f(x)\,dx = \int_a^{p_1} f(x)\,dx + \int_{p_1}^{p_2} f(x)\,dx + \cdots + \int_{p_{m-1}}^{p_m} f(x)\,dx + \int_{p_m}^{b} f(x)\,dx.$$

§7.2 # 1. Clearly

$$\left[\int_0^1 f(t)^n\,dt\right]^{1/n} \le \left[\int_0^1 M^n\,dt\right]^{1/n} = [M^n]^{1/n} = M.$$

For the opposite inequality, let $p \in [0,1]$ be a point where $f(p) = M$. Let $\varepsilon > 0$. Choose $\delta > 0$ so that $f(t) \ge M - \varepsilon$ on $[p-\delta, p+\delta]$. Then

$$\left[\int_0^1 f(t)^n\,dt\right]^{1/n} \ge \left[\int_{p-\delta}^{p+\delta} [M-\varepsilon]^n\,dt\right]^{1/n}$$

$$= (2\delta)^{1/n}\cdot[M-\varepsilon] \to M-\varepsilon \quad \text{as } n \to \infty.$$

Since this is true for every $\varepsilon > 0$, we have

$$\lim_{n\to\infty}\left[\int_0^1 f(t)^n\right]^{1/n} \ge M.$$

So $\lim_{n\to\infty}\left[\int_0^1 f(t)^n\right]^{1/n} = M.$

4. $$\int_8^6 f(x)\,dx = \int_8^3 f(x)\,dx + \int_3^6 f(x)\,dx$$

$$= -5 + 2 = -3.$$

5. Since g is continuous on $[0,1]$, g is bounded. Say that $|g(x)| \le M$ for all x. Then

$$|Tf| = \left|\int_0^1 f(x)g(x)\,dx\right| \le \int_0^1 |f(x)|\,|g(x)|\,dx$$

$$\le \int_0^1 |f(x)| \cdot M\,dx = M \cdot \int_0^1 |f(x)|\,dx.$$

9. $$\int_0^1 \frac{\cos(2r) - \cos(r)}{r}\,dr = \int_0^1 \frac{\left(1 - (2r)^2 + \cdots\right) - \cos\left(1 - r^2 + \cdots\right)}{r}\,dr$$

$$= \int_0^1 \frac{-r^2 + \text{higher order terms}}{r}\,dr$$

$$= \int_0^1 -r + \text{higher order terms}\,dr$$

and this integral exists.

$$\int_0^{1/2} \frac{\cos(2r) - \cos r}{r}\,dr = \int_1^{1/2} \frac{\cos 2r}{r}\,dr - \int_1^{1/2} \frac{\cos r}{r}\,dr$$

$$(\text{parts}) = \left[\frac{\frac{1}{2}\sin 2r}{r}\right]_1^{1/2} + \int_1^{1/2} \frac{\frac{1}{2}\sin 2r}{r^2}\,dr$$

$$- \left[\frac{\sin r}{r}\right]_1^{1/2} + \int_1^{1/2} \frac{\sin r}{r^2}\,dr$$

As $\eta \to 0$ This tends to

$$-\frac{1}{2}\sin 2 + \int_1^\infty \frac{\frac{1}{2}\sin 2v}{v^2}\,dv$$

$$+\sin 1 - \int_1^\infty \frac{\sin v}{v^2}\,dv,$$

Both These integrals converge.

§7.3 #1. $\lim_{t\to a^+} \left| \frac{F(t)-F(a)}{t-a} - f(a) \right|$

$$= \lim_{t\to a^+} \left| \frac{\int_a^t f(s)\,ds - 0}{t-a} - \frac{\int_a^t f(a)\,ds}{t-a} \right|$$

$$= \lim_{t\to a^+} \left| \frac{\int_a^t [f(s)-f(a)]\,ds}{t-a} \right|$$

$$\le \lim_{t\to a^+} \frac{\int_a^t |f(s)-f(a)|\,ds}{|t-a|}. \qquad (*)$$

Let $\varepsilon > 0$. Choose $\delta > 0$ s.t. $|s-a| < \delta \Rightarrow$ $|f(s)-f(a)| < \varepsilon$. For $|t-a| < \delta$ we have then that

$$(*) \le \frac{\int_a^t \varepsilon\,ds}{|t-a|} = \varepsilon.$$

So $\lim_{t\to a^+} \frac{F(t)-F(a)}{t-a} = f(a)$.

The result at b is proved similarly.

4. Let $f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ -1 & \text{if } x \text{ is irrational} \end{cases}$

on $[0, 1]$. This is a version of the Dirichlet function, and it is <u>not</u> Riemann integrable. But $f^2(x) \equiv 1$ <u>is</u> Riemann integrable.

6. $f(x) = \int_0^{\cos x^2} e^{\sin t}\, dt$

Let $h(x) = \int_0^x e^{\sin t}\, dt$

and $g(x) = \cos x^2$.

Then $f(x) = h \circ g(x)$. So

$f'(x) = h'(g(x)) \cdot g'(x)$

$= e^{\sin(\cos x^2)} \cdot (-\sin x^2) \cdot 2x$.

10. Let $[a, b] = [0, 1]$. Let $f(x) = x$, $g(x) = x$. Then

$\int_0^1 f(x) \cdot g(x)\, dx = \int_0^1 x^2\, dx = \frac{1}{3}$.

But $\int_0^1 f(x)\, dx = \int_0^1 x\, dx = \frac{1}{2}$

$\int_0^1 g(x)\, dx = \int_0^1 x\, dx = \frac{1}{2}$.

And $\frac{1}{3} \neq \frac{1}{2} \cdot \frac{1}{2}$.

§7.4 #1. Any Δ_j that occurs strictly inside an interval $[k, k+1)$ will be just like the Δ_j for $\alpha(x) = x$. In other words, just like the ordinary Riemann integral. So the only Δ_j that make a difference are for intervals that contain integers. Thus the value of the integral is

$$\int_2^6 t^2\,dt + t^2\Big]_{t=3} + t^2\Big]_{t=4} + t^2\Big]_{t=5}$$

$$= \frac{t^3}{3}\Big]_2^6 + 16 + 25 = \frac{216}{3} - \frac{8}{3} + 41 = 113 - \frac{8}{3} = \frac{331}{3}.$$

2. $$\int_0^5 x\,d\gamma(x) = \int_0^5 x\,dx - \int_0^5 x\,d\alpha(x)$$

$$= \frac{x^2}{2}\Big]_0^5 - x\Big|_{x=1} - x\Big|_{x=2} - x\Big|_{x=3} - x\Big|_{x=4}$$

$$= \frac{25}{2} - 1 - 2 - 3 - 4$$

$$= \frac{25}{2} - 10 = \frac{5}{2}.$$

5. $$\int_0^3 H(x)\,d\alpha(x) = \int_0^3 x^2\,d\alpha(x) = x^2\Big|_{x=1} + x^2\Big|_{x=2}$$

$$= 1 + 4 = 5.$$

9. $$\int_0^\pi x^2 \, d\alpha(x) = \int_0^\pi x^2 \cdot \alpha'(x) \, dx$$

$$= \int_0^\pi x^2 \cdot 3x^2 \, dx = 3 \int_0^\pi x^4 \, dx$$

$$= \frac{3x^5}{5} \Big]_0^\pi = \frac{3\pi^5}{5}.$$