# SOLUTIONS TO MATH 4111 HW3

## Section 2.3

1. Let $\{a_j\} = \left\{\frac{1}{j}\right\}_{j=1}^{\infty}$.

Then $\limsup a_j = 0$,

$\sup\{a_j\} = 1$.

Let $\{b_j\} = \left\{-\frac{1}{j}\right\}_{j=1}^{\infty}$.

Then $\liminf b_j = 0$

$\inf\{b_j\} = -1$

3. $\limsup a_j = -\liminf -a_j$

$\liminf a_j = -\limsup -a_j$.

5. Let $\alpha_j = \begin{cases} b - \frac{1}{j} & \text{if } j \text{ is odd} \\ a + \frac{1}{j} & \text{if } j \text{ is even.} \end{cases}$

7. $B_j = \sup\{a_j + b_j, a_{j+1} + b_{j+1}, \dots\}$

$\leq \sup\{a_j, a_{j+1}, \dots\} + \sup\{b_j, b_{j+1}, \dots\}$

$= B_j^0 + B_j^1$

Then $B_j \to \limsup(a_j + b_j)$

$B_j^0 \to \limsup a_j$

$B_j^1 \to \limsup b_j$

So $\limsup(a_j + b_j) \leq \limsup a_j + \limsup b_j$

Similarly for the liminfs.

For $\limsup a_j b_j$ look at

$B_j = \sup\{a_j b_j, a_{j+1} b_{j+1}, \dots\}$

$\le \sup\{a_j, a_{j+1}, \dots\} \sup\{b_j, b_{j+1}, \dots\}$.

$\equiv B_j^0 \cdot B_j^1$ as long as $a_j \ge 0$, $b_j \ge 0$.

Letting $j \to \infty$ gives

$\limsup(a_j b_j) \le \limsup a_j \cdot \limsup b_j$.

Similar for liminfs.

Section 2.4

2. $\left(\frac{1}{j}\right)^{1/j} = \frac{1}{j^{1/j}}$. But $j^{1/j} \to 1$. So

$\left(\frac{1}{j}\right)^{1/j} \to \frac{1}{1} = 1$.

3. $\left\{\frac{j^j}{(2j)!}\right\}_{j=2}^{\infty}$

$$\frac{j^j}{(2j)!} \le \frac{1}{(1 \cdot 2 \cdots j)} \cdot \frac{j^j}{(j+1)(j+2)\cdots 2j}$$

$$\le \frac{1}{1 \cdot 2 \cdots j}.$$

This clearly tends to 0.

9. $a_j = \left(1+\frac{1}{1^2}\right)\left(1+\frac{1}{2^2}\right)\left(1+\frac{1}{3^2}\right)\cdots\left(1+\frac{1}{j^2}\right)$

$= 1 + \sum_{l=1}^{j} \frac{1}{l^2} +$ higher order terms.

This $\nearrow$ series converges. So the sequence of products converges.

Section 3.1

12) $\sum_{j=1}^{\infty} \frac{(2^j)^2}{j!}$

$\frac{a_{j+1}}{a_j} = \frac{(2^{j+1})^2/(j+1)!}{(2^j)^2/j!} = \frac{4}{j+1} \to 0$

So series converges.

c) $\sum_{j=1}^{\infty} \frac{j!}{j^j}$.

$\frac{a_{j+1}}{a_j} = \frac{(j+1)!/(j+1)^{j+1}}{j!/j^j} = \frac{(j+1)\,j^j}{(j+1)^{j+1}} = \frac{j^j}{(j+1)^j} = \frac{1}{(1+\frac{1}{j})^j} \to \frac{1}{e} < 1.$

So series converges.

f) $\sum_{j=1}^{\infty} \frac{2j-1}{3j^3-2}$

$\frac{2j-1}{3j^3-2} < \frac{1}{4}\frac{j}{j^3} = \frac{1}{4}\frac{1}{j^2}$ for $j$ large.

So series converges.

3. $b_j > 0$, $\sum b_j$ converges. Then, for $j$ large, $b_j < \frac{1}{2}$. So

$$\frac{1}{1+b_j} > \frac{1}{3/2} = \frac{2}{3}.$$

Hence the terms of $\sum \frac{1}{1+b_j}$ do not tend to 0. The series diverges.

6. $b_j > 0$, $\sum b_j$ converges.

$$\frac{b_j}{1+b_j} < b_j. \text{ So } \sum \frac{b_j}{1+b_j} \text{ converges.}$$

9. $b_j > 0$, $\sum b_j$ converges

$$2^{-j} b_j \leq b_j.$$

So $\sum 2^{-j} b_j$ converges.

Section 3.2

2. If $j$ is odd then

$$a_j^{1/j} = \left(\frac{1}{3^{(j+1)/2}}\right)^{\frac{1}{j}} = \frac{1}{3^{\frac{j+1}{2j}}} \to \frac{1}{3} < 1$$

If $j$ is even then

$$a_j^{1/j} = \left(\frac{1}{5^{j/2}}\right)^{1/j} = \frac{1}{5^{1/2}} < 1.$$

So series passes root test and converges.

For the ratio test, note that

$$\frac{1/3^{2j}}{1/5^{2j-1}} = \frac{5^{2j-1}}{3^{2j}} = \left(\frac{5}{3}\right)^{2j} \cdot \frac{1}{5} \to +\infty$$

while

$$\frac{1/5^{2j+1}}{1/3^{2j+1}} = \left(\frac{3}{5}\right)^{2j+1} \to 0.$$ So ratio test inconclusive.

4. Assume $a_j \to \ell$. Set $\sigma_j = \dfrac{a_1 + a_2 + \cdots + a_j}{j}$.

Given $\varepsilon > 0$, choose $n_0 \in \mathbb{N}$ such that

$|a_j - \ell| < \frac{\varepsilon}{2}$ for $j > n_0$.

The $a_j$ are bounded, so we can choose $N > n_0$ such that

$$\frac{1}{N} \sum_{j=1}^{n_0} |a_j - \ell| < \frac{\varepsilon}{2}.$$

Let $n > N$. Then

$$|\ell - \sigma_n| = \left|\frac{1}{n}\sum_{j=1}^{n} \ell - a_j\right| \le \frac{1}{n}\sum_{\ell=0}^{n_0} |a_j - \ell| + \frac{1}{n}\sum_{\ell=n_0+1}^{n} |a_j - \ell|$$

$$< \frac{\varepsilon}{2} + \frac{n-n_0}{n+1}\frac{\varepsilon}{2}$$

$$< \varepsilon.$$

Thus $\sigma_n \to \ell$.

6. $\frac{1}{C} \le \frac{a_j}{b_j} \le C$. So (1) $b_j \le C a_j$ and (2) $a_j \le C b_j$.

If $\sum a_j$ converges then (1) and comparison test say that $\sum b_j$ converges. If $\sum b_j$ converges then (2) and comparison test say that $\sum a_j$ converges.

Section 3.3.

1. $\frac{1}{2} > b_j > 0 \ \forall j$ and $\sum b_j$ converges.

Then $0 < \frac{b_j}{1-b_j} \leq \frac{b_j}{1/2} = 2b_j$. Then

$\sum \frac{b_j}{1-b_j}$ converges by comparison test.

4. $\sum_{j=2}^{\infty} \frac{1}{j^k (\log j)^\ell}$.

Use the integral test to see that

$\sum_{j=2}^{\infty} \frac{1}{j (\log j)^\ell}$ converges for $\ell > 1$.

It diverges for $\ell = 1$.

Trivially, $\sum_{j=2}^{\infty} \frac{1}{j^k} \frac{1}{(\log j)^\ell}$

converges for $k > 1$ and any $\ell$.

7. $\gamma_j > 0$, $\sum_{j=1}^{\infty} \gamma_j$ converges. Then, by Cauchy–Schwarz,

$$\sum (\gamma_j)^{1/2} \frac{1}{j^\alpha} \leq \left(\sum \gamma_j\right)^{\frac{1}{2}} \left(\sum_j \frac{1}{j^{2\alpha}}\right)^{1/2}.$$

$\sum \gamma_j$ converges and $\sum_j \frac{1}{j^{2\alpha}}$ converges since

$2\alpha > 1$.

Section 3.4

1. $S_{k,N} = \frac{1}{k+1}\left[(N+1)^{k+1} - 1 - N - \frac{(k+1)\cdot k}{2} S_{k-1,N} - \cdots - \frac{(k+1)k}{2} S_{2,N} - (k+1) S_{1,N}\right]$. (*)

We know that

$$S_{1,N} = \frac{N(N+1)}{2}.$$

So, by (*),

$$S_{2,N} = \frac{1}{3}\left[(N+1)^3 - 1 - N - \frac{3\cdot 2}{2}\cdot\frac{N(N-1)}{2}\right]$$

$$= \frac{1}{3}\left[N^3 + 3N^2 + 3N + 1 - 1 - N - 3\,\frac{N(N-1)}{2}\right]$$

$$= \frac{1}{3}\left[\frac{2N^3 + 6N^2 + 4N - 3N^2 - 3N}{2}\right]$$

$$= \frac{2N^3 + 3N^2 + N}{6} = \frac{N(N+1)(2N+1)}{6}$$

$$S_{3,N} = \frac{1}{4}\left[(N+1)^4 - 1 - N - \frac{4\cdot 3}{2}\cdot\frac{N(N+1)(2N+1)}{6} - (4)\frac{N(N+1)}{2}\right]$$

$$= \frac{1}{4}\left[N^4 + 4N^3 + 6N^2 + 4N + 1 - 1 - N - N(N+1)(2N+1) - 2N(N+1)\right]$$

$$= \frac{1}{4}\left[N^4 + 4N^3 + 6N^2 + 4N + 1 - 1 - N - 2N^3 - 3N^2 - N - 2N^2 - 2N\right]$$

$$= \left[N^4 + 2N^3 + N^2\right]/4 = \frac{N^2(N^2+2N+1)}{4} = \frac{N^2(N+1)^2}{4}.$$

That's enough of that.

3. $\sum_{j=1}^{\infty} \frac{1}{(\ln j)^k}$.

Note that $\frac{1}{(\ln j)^k} > \frac{1}{j}$ for any $k > 0$.

So $\sum \frac{1}{(\ln j)^k}$ diverges.

5. $\sum_j \exp(p(j))$

$$p(j) = a_0 + a_1 j + \cdots + a_k j^k.$$

Clearly $a_k j^k$ is the most important term.
If $a_k > 0$ then the series diverges because the summands don't tend to 0.
If $a_k < 0$ then the series is comparable to a geometric series so converges.

Section 3.5

1. $\sum_{j=1}^{\infty} \frac{1}{j^3}^{a_j} \cdot \sum_{j=1}^{\infty} \frac{1}{j^4}^{b_j} = \sum_{k=1}^{\infty} c_k$

where $c_k = \sum_{j=0}^{k} a_j b_{k-j}$

So

$$c_0 = a_0 b_0 = 1 \cdot 1 = 1$$

$$c_1 = a_0 b_1 + a_1 b_0 = 1 \cdot \frac{1}{1^4} + \frac{1}{1^3} \cdot 1 = 2$$

$$c_2 = a_0 b_2 + a_1 b_1 + a_2 b_0 = 1 \cdot \frac{1}{2^4} + \frac{1}{1^3} \cdot \frac{1}{1^4} + \frac{1}{2^3} \cdot 1$$

$$= \frac{1}{16} + 1 + \frac{1}{8} = \frac{3}{16}$$

$$c_3 = a_0 b_3 + a_1 b_2 + a_2 b_1 + a_3 b_0$$

$$= 1 \cdot \frac{1}{3^4} + \frac{1}{1^3} \cdot \frac{1}{2^4} + \frac{1}{2^3} \cdot \frac{1}{1^4} + \frac{1}{3^3} \cdot 1$$

$$= \frac{1}{81} + \frac{1}{16} + \frac{1}{8} + \frac{1}{27}$$

$$= \frac{307}{1296}$$

In general, $c_j < (j+1) \cdot \frac{1}{2^j}$.

So the product series will certainly converge.

5. For $j$ large enough $b_j < 1$. So

$$0 < a_j b_j < a_j.$$

Thus $\sum a_j b_j$ converges by the comparison test.

6. Let $a_j = \frac{1}{j^2}$ and $b_j = \frac{1}{j^2}$. Then

$$\sum \frac{a_j}{b_j} = \sum 1,$$ which diverges.