MATH 4111

Solutions to Final Exam

1. If $\mathbb{C}$ were an ordered field then, since $i \neq 0$, then either $i > 0$ or $i < 0$.

If $i > 0$ Then $i \cdot i > 0$ or $-1 > 0$ which is absurd.

If $i < 0$ then $-i > 0$ so $(-i)\cdot(-i) > 0$ or $-1 > 0$ which is absurd.

So $\mathbb{C}$ cannot be an ordered field.

2. Let $\varepsilon > 0$. Then $\alpha - \varepsilon$ is not an upper bound for S. Hence $\exists\, s \in S$ such that $\alpha - \varepsilon < s \leq \alpha$.

3. $\dfrac{\sin^2 j}{j} = \dfrac{1 - \cos 2j}{j}$. The series

$\displaystyle\sum_{j=1}^{\infty} \frac{\cos 2j}{j}$ converges by a summation-by-parts argument like Example 3.35 in the text. And the series $\displaystyle\sum_j \frac{1}{j}$ diverges. Hence

$\displaystyle\sum \frac{\sin^2 j}{j}$ diverges.

4. Let $a_j = \left(\frac{1}{j}\right)^{\frac{1}{j}}$. Then let $b_j = \log a_j = \frac{1}{j}\log\frac{1}{j}$

$= \dfrac{\log(1/j)}{j}$. Then numerator tends to $-\infty$ and denominator tends to $+\infty$. So l'Hôpital's Rule applies.

$$\lim_{x\to\infty} \frac{\log(1/x)}{x} = \lim_{x\to\infty} \frac{-1/x}{1} = 0.$$

So $\lim_{j\to\infty} b_j = 0$ hence $\lim_{j\to\infty} a_j = 1$.

5. Use the integral test:

$$\int_3^\infty \frac{1}{x\log^3 x}\,dx = -\frac{1}{2}\frac{1}{\log^2 x}\Big]_3^\infty = \frac{1}{2}\frac{1}{\log^2 3} < \infty.$$

So the series converges.

6. The series $\sum \frac{1}{j^3}$ converges. And the series

$$\sum \frac{1}{j^3}\cdot j = \sum \frac{1}{j^2}$$ also converges.

But the series $\sum \frac{1}{j^2}$ converges yet $\sum \frac{1}{j^2}\cdot j = \sum \frac{1}{j}$ diverges.

Assume $f(0) = 1$,

7. Since $0 < f'(x) \le f(x)$ then

$$\frac{f'(x)}{f(x)} \le 1$$

$$\int_0^t \frac{f'(x)}{f(x)}\,dx \le \int_0^t 1\,dx$$

$$\log f(t) \le t$$

$$f(t) \le e^t.$$

8. If $f''(x) \geq c > 0$, then

$$\int_0^t f''(x)\,dx \geq \int_0^t c\,dx$$

$$f'(t) - f'(0) \geq ct$$

$$\int_0^w f'(t) - f'(0)\,dt \geq \int_0^w ct\,dt$$

$$[f(w) - f'(0)w] - [f(0)] \geq \frac{cw^2}{2} + 0$$

$$f(w) \geq \frac{cw^2}{2} + f'(0)w + f(0) + 0$$

So f grows parabolically.

9. Let $E_j = [1/2_j,\ 1 - 1/2_j]$ for $j = 2, 3, \ldots$.

Then each E_j is closed but

$$\bigcup_{j=2}^{\infty} E_j = (0,1) \quad \text{is open}$$

Let $F_j = [0, 1 - \frac{1}{2_j}]$. Then each E_j is closed but

$$\bigcup_{j=1}^{\infty} E_j = [0,1) \quad \text{is half-open.}$$

Let $K_j = [0, j]$. Then each K_j is closed and

$$\bigcup_{j=1}^{\infty} K_j = [0,\infty) \quad \text{is closed,}$$

10. Let $p_N = \sum_{j=0}^{N} (-1)^j \frac{x^{2j}}{(2j)!}$. Then each p_N is a polynomial but $p_N \to \cos x$ which is <u>not</u> a polynomial.

11. Let $\varepsilon > 0$. Let $x \in [0,1]$. There exists an index n so large that $|f_n(x) - f(x)| < \varepsilon/2$. By continuity, $|f_n(t) - f(t)| < \varepsilon/2$ for t in a small open interval I_x centered at x. The intervals $\{I_x\}$ cover $[0,1]$. By compactness, there is a finite subcover $I_{x_1}, \ldots, I_{x_k}$. Let f_{n_j} be the function associated to I_{x_j}. Let $N = \max\{n_1, \ldots, n_k\}$. Then, since $f_1 \geq f_2 \geq f_3 \geq \ldots \to f$, $|f_n(x) - f(x)| < \varepsilon$ for all $n > N$ and all $x \in [0,1]$. That is uniform convergence.

12. We may as well take a partition with mesh less than 1. The only terms of the Riemann-Stieltjes sum that are non-zero are the intervals that contain 1, 2, 3, 4, 5. For each of these, $\Delta\alpha_j = 1$. So the value of the integral is

$$1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 1 + 4 + 9 + 16 + 25 = 55.$$

13. $$\int_0^2 f(x)\,dg(x) = \int_0^2 f(x)g'(x)\,dx = \int_0^1 \sin x \cdot 2x\,dx$$

$$= 2\left[x(-\cos x)\Big]_0^1 + \int_0^1 \cos x\,dx\right] = -2\cos 1 + 2\sin x\Big]_0^1 = -2\cos 1 + 2\sin 1$$

14. $\left| \sum_{j=M}^{N} \frac{\sin jx}{j^2} \right| \leq \sum_{j=M}^{N} \frac{1}{j^2} < \varepsilon$ if M, N large enough. So the series is uniformly Cauchy. The uniform limit of continuous functions is continuous.