Math 4111
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Figure: This is your instructor.

More on Uniform Convergence

In general, limits do not commute. Since the integral is defined with a limit, and since we saw in the last section that integrals do not always respect limits of functions, we know some concrete instances of non-commutation of limits. The fact that continuity is defined with a limit, and that the limit of continuous functions need not be continuous, gives even more examples of situations in which limits do not commute. Let us now turn to a situation in which limits *do* commute:

Theorem: *Fix a set S and a point $s \in S$. Assume that the functions f_j converge uniformly on the domain $S \setminus \{s\}$ to a limit function f. Suppose that each function $f_j(x)$ has a limit as $x \to s$. Then f itself has a limit as $x \to s$ and*

$$\lim_{x \to s} f(x) = \lim_{j \to \infty} \lim_{x \to s} f_j(x) .$$

Because of the way that f is defined, we may rewrite this conclusion as

$$\lim_{x \to s} \lim_{j \to \infty} f_j(x) = \lim_{j \to \infty} \lim_{x \to s} f_j(x) .$$

In other words, the limits $\lim_{x \to s}$ and $\lim_{j \to \infty}$ commute.

Proof: Let $\alpha_j = \lim_{x \to s} f_j(x)$. Let $\epsilon > 0$. There is a number $N > 0$ (independent of $x \in S \setminus \{s\}$) such that $j > N$ implies that $|f_j(x) - f(x)| < \epsilon/4$. Fix $j, k > N$. Choose $\delta > 0$ such that $0 < |x - s| < \delta$ implies both that $|f_j(x) - \alpha_j| < \epsilon/4$ and $|f_k(x) - \alpha_k| < \epsilon/4$. Then

$$|\alpha_j - \alpha_k| \leq |\alpha_j - f_j(x)| + |f_j(x) - f(x)| + |f(x) - f_k(x)| + |f_k(x) - \alpha_k| \ .$$

The first and last expressions are less than $\epsilon/4$ by the choice of x. The middle two expressions are less than $\epsilon/4$ by the choice of N (and therefore of j and k). We conclude that the sequence α_j is Cauchy. Let α be the limit of that sequence.

Letting $k \to \infty$ in the inequality

$$|\alpha_j - \alpha_k| < \epsilon$$

that we obtained above yields

$$|\alpha_j - \alpha| \leq \epsilon$$

for $j > N$. Now, with δ as above and $0 < |x - s| < \delta$, we have

$$|f(x) - \alpha| \leq |f(x) - f_j(x)| + |f_j(x) - \alpha_j| + |\alpha_j - \alpha| \ .$$

By the choices we have made, the first term is less than $\epsilon/4$, the second is less than $\epsilon/4$, and the third is less than or equal to ϵ. Altogether, if $0 < |x - s| < \delta$ then $|f(x) - \alpha| < 2\epsilon$. This is the desired conclusion. $\square$

Example: Consider the example

$$f_j(x) = x^j$$

on the interval $[0, 1]$. We see that

$$\lim_{j\to\infty} f_j(x) = 0 \equiv f(x)$$

for $0 \leq x < 1$. Thus

$$\lim_{x\to 1^-} f(x) = 0\,.$$

But

$$\lim_{j\to\infty} \lim_{x\to 1^-} f_j(x) = \lim_{j\to\infty} 1 = 1\,.$$

Thus the two dual limits in the theorem are unequal in this example. But of course the functions f_j do not converge uniformly. □

Parallel with our notion of Cauchy sequence of numbers, we have a concept of Cauchy sequence of functions in the uniform sense:

Definition: A sequence of functions f_j on a domain S is called *a uniformly Cauchy sequence* if, for each $\epsilon > 0$, there is an $N > 0$ such that, if $j, k > N$, then

$$|f_j(x) - f_k(x)| < \epsilon \quad \forall x \in S\,.$$

The key point for "uniformly Cauchy" sequence of functions is that the choice of N does not depend on x.

Proposition: *A sequence of functions f_j is uniformly Cauchy on a domain S if and only if the sequence converges uniformly to a limit function f on the domain S.*

Proof: The proof is straightforward and is assigned as an exercise. □

We will use the last two results in our study of the limits of differentiable functions. First we consider an example.

Example: Define the function

$$f_j(x) = \begin{cases} 0 & \text{if} \quad x \leq 0 \\ jx^2 & \text{if} \quad 0 < x \leq 1/(2j) \\ x - 1/(4j) & \text{if} \quad 1/(2j) < x < \infty \end{cases}$$

We leave it as an exercise for you to check that the functions f_j converge uniformly on the entire real line to the function

$$f(x) = \begin{cases} 0 & \text{if} \quad x \leq 0 \\ x & \text{if} \quad x > 0 \end{cases}$$

(draw a sketch to help you see this).

Notice that each of the functions f_j is continuously differentiable on the entire real line, but f is not differentiable at 0. □

It turns out that we must strengthen our convergence hypotheses if we want the limit process to respect differentiation. The basic result is this:

Theorem: *Suppose that a sequence f_j of differentiable functions on an open interval I converges pointwise to a limit function f. Suppose further that the sequence f_j' converges uniformly on I to a limit function g. Then the limit function f is differentiable on I and $f'(x) = g(x)$ for all $x \in I$.*

Proof: Let $\epsilon > 0$. The sequence $\{f_j'\}$ is uniformly Cauchy. Therefore we may choose N so large that $j, k > N$ implies that

$$\left|f_j'(x) - f_k'(x)\right| < \frac{\epsilon}{2} \quad \forall x \in I\,.$$

Fix a point $P \in I$. Define

$$\mu_j(x) = \frac{f_j(x) - f_j(P)}{x - P}$$

for $x \in I, x \neq P$. It is our intention to apply the theorem above to the functions μ_j.

First notice that, for each j, we have

$$\lim_{x \to P} \mu_j(x) = f_j'(P)\,.$$

Thus

$$\lim_{j \to \infty} \lim_{x \to P} \mu_j(x) = \lim_{j \to \infty} f_j'(P) = g(P)\,.$$

That calculates the limits in one order.

On the other hand,

$$\lim_{j\to\infty} \mu_j(x) = \frac{f(x) - f(P)}{x - P} \equiv \mu(x)$$

for $x \in I \setminus \{P\}$. If we can show that this convergence is uniform then the theorem applies and we may conclude that

$$\lim_{x\to P} \mu(x) = \lim_{j\to\infty} \lim_{x\to P} \mu_j(x) = \lim_{j\to\infty} f_j'(P) = g(P)\,.$$

But this just says that f is differentiable at P and the derivative equals g. That is the desired result.

To verify the uniform convergence of the μ_j, we apply the Mean Value Theorem to the function $f_j - f_k$. For $x \neq P$ we have

$$\begin{aligned} |\mu_j(x) - \mu_k(x)| &= \frac{1}{|x-P|} \cdot |(f_j(x) - f_k(x)) - (f_j(P) - f_k(P))| \\ &= \frac{1}{|x-P|} \cdot |x-P| \cdot |(f_j - f_k)'(\xi)| \\ &= |(f_j - f_k)'(\xi)| \end{aligned}$$

for some ξ between x and P. But the displayed line at the beginning of the proof guarantees that the last line does not exceed $\epsilon/2$. That shows that the μ_j converge uniformly and concludes the proof. $\square$

Remark: A little additional effort shows that we need only assume in the theorem that the functions f_j converge at a single point x_0 in the domain. One of the exercises asks you to prove this assertion.

Notice further that, if we make the additional assumption that each of the functions f_j' is continuous, then the proof of the theorem becomes much easier. For then

$$f_j(x) = f_j(x_0) + \int_{x_0}^{x} f_j'(t)\,dt$$

by the Fundamental Theorem of Calculus. The hypothesis that the f_j' converge uniformly then implies that the integrals converge to

$$\int_{x_0}^{x} g(t)\,dt\,.$$

The hypothesis that the functions f_j converge at x_0 then allows us to conclude that the sequence $f_j(x)$ converges for every x to $f(x)$ and

$$f(x) = f(x_0) + \int_{x_0}^{x} g(t)\,dt\,.$$

The Fundamental Theorem of Calculus then yields that $f' = g$ as desired.

Example: Consider the sequence of functions $f_j(x) = j^{-1/2}\sin(jx)$. This sequence converges uniformly to the identically zero function $f(x) \equiv 0$. But $f_j' = j^{1/2}\cos(jx)$ does not converge at any point.

We can sum up this result by saying that

$$\lim_{j\to\infty} \frac{d}{dx} f_j(x) \neq \frac{d}{dx} \lim_{j\to\infty} f_j(x)\,. \qquad \square$$