Math 4111
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Figure: This is your instructor.

Partial Sums and Pointwise Convergence for Series of Functions

A *sequence of functions* is usually written

$$f_1, f_2, \ldots \quad \text{or} \quad \{f_j\}_{j=1}^{\infty} .$$

We will generally assume that the functions f_j all have the same domain S.

Definition: A sequence of functions $\{f_j\}_{j=1}^{\infty}$ with domain $S \subseteq \mathbb{R}$ is said to *converge pointwise* to a limit function f on S if, for each $x \in S$, the sequence of numbers $\{f_j(x)\}$ converges to $f(x)$.

Example: Define $f_j(x) = x^j$ with domain $S = \{x : 0 \leq x \leq 1\}$. If $0 \leq x < 1$ then $f_j(x) \to 0$. However, $f_j(1) \to 1$. Therefore the sequence f_j converges pointwise to the function

$$f(x) = \begin{cases} 0 & \text{if} \quad 0 \leq x < 1 \\ 1 & \text{if} \quad x = 1 \end{cases}$$

See the figure. We see that, even though the f_j are each continuous, the limit function f is not. □

Here are some of the basic questions that we must ask about a sequence of functions f_j that converges to a function f on a domain S:

(1) If the functions f_j are continuous, then is f continuous?

(2) If the functions f_j are integrable on an interval I, then is f integrable on I?

(3) If f is integrable on I, then does the sequence $\int_I f_j(x)\,dx$ converge to $\int_I f(x)\,dx$?

(4) If the functions f_j are differentiable, then is f differentiable?

(5) If f is differentiable, then does the sequence f_j' converge to f'?

We see from the last example that the answer to the first question is "no": Each of the f_j is continuous but f is not. It turns out that, in order to obtain a favorable answer to our questions, we must consider a stricter notion of convergence of functions. This motivates the next definition.

Definition: Let f_j be a sequence of functions on a domain S. We say that the functions f_j *converge uniformly* to f on S if, given $\epsilon > 0$, there is an $N > 0$ such that, for any $j > N$ and any $x \in S$, it holds that $|f_j(x) - f(x)| < \epsilon$.

Notice that the special feature of uniform convergence is that the rate at which $f_j(x)$ converges is independent of $x \in S$. In the first example, $f_j(x)$ is converging very rapidly to zero for x near zero but arbitrarily slowly to zero for x near 1—again see the figure. In the next example we shall prove this assertion rigorously:

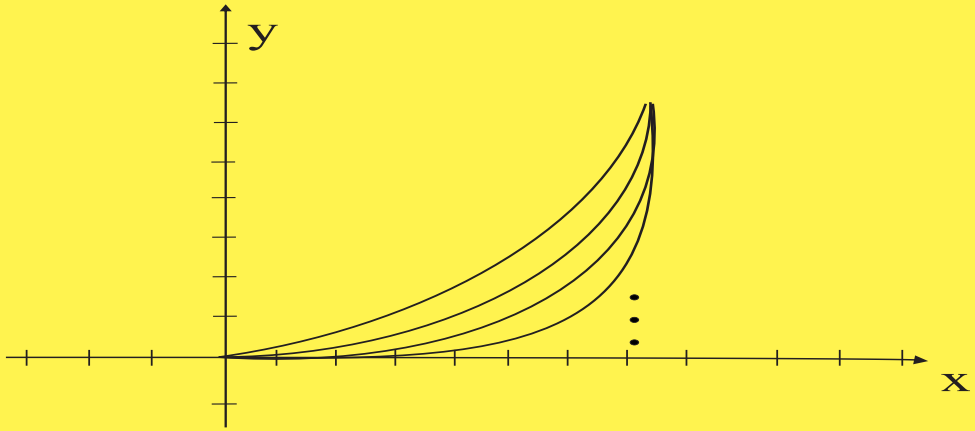


Figure: The sequence $\{x^j\}$.

Example: The sequence $f_j(x) = x^j$ does *not* converge uniformly to the limit function

$$f(x) = \begin{cases} 0 & \text{if} \quad 0 \le x < 1 \\ 1 & \text{if} \quad x = 1 \end{cases}$$

on the domain $S = [0, 1]$. In fact it does not even do so on the smaller domain $[0, 1)$. To see this notice that, no matter how large j is, we have by the Mean Value Theorem that

$$f_j(1) - f_j(1 - 1/(2j)) = \frac{1}{2j} \cdot f_j'(\xi)$$

for some ξ between $1 - 1/(2j)$ and 1.

But $f_j'(x) = j \cdot x^{j-1}$ hence $|f_j'(\xi)| < j$ and we conclude that

$$|f_j(1) - f_j(1 - 1/(2j))| < \frac{1}{2}$$

or

$$f_j(1 - 1/(2j)) > f_j(1) - \frac{1}{2} = \frac{1}{2}\,.$$

In conclusion, no matter how large j, there will be values of x (namely, $x = 1 - 1/(2j)$) at which $f_j(x)$ is at least distance $1/2$ from the limit 0. We conclude that the convergence is not uniform. □

Theorem: *If f_j are continuous functions on a set S that converge uniformly on S to a function f then f is also continuous.*

Proof: Let $\epsilon > 0$. Choose an integer N so large that, if $j > N$, then $|f_j(x) - f(x)| < \epsilon/3$ for all $x \in S$. Fix $P \in S$. Choose $\delta > 0$ so small that if $|x - P| < \delta$ then $|f_{N+1}(x) - f_{N+1}(P)| < \epsilon/3$. For such x we have

$$\begin{aligned} |f(x) - f(P)| &\leq |f(x) - f_{N+1}(x)| + |f_{N+1}(x) - f_{N+1}(P)| \\ &\quad + |f_{N+1}(P) - f(P)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \end{aligned}$$

by the way that we chose N and δ. But the last line sums to ϵ, proving that f is continuous at P. Since $P \in S$ was chosen arbitrarily, we are done. $\square$

Example: Define functions

$$f_j(x) = \begin{cases} 0 & \text{if} \quad x = 0 \\ j & \text{if} \quad 0 < x \le 1/j \\ 0 & \text{if} \quad 1/j < x \le 1 \end{cases}$$

for $j = 2, 3, \ldots$. Then $\lim_{j\to\infty} f_j(x) = 0 \equiv f(x)$ for all x in the interval $I = [0, 1]$. However

$$\int_0^1 f_j(x)\,dx = \int_0^{1/j} j\,dx = 1$$

for every j. Thus the f_j converge to the integrable limit function $f(x) \equiv 0$ (with integral 0), but their integrals do not converge to the integral of f.

Of course the f_j do not converge uniformly. □

Example: Let $q_1, q_2, \ldots$ be an enumeration of the rationals in the interval $I = [0, 1]$. Define functions

$$f_j(x) = \begin{cases} 1 & \text{if} \quad x \in \{q_1, q_2, \ldots, q_j\} \\ 0 & \text{if} \quad x \notin \{q_1, q_2, \ldots, q_j\} \end{cases}$$

Then the functions f_j converge pointwise to the Dirichlet function f which is equal to 1 on the rationals and 0 on the irrationals. Each of the functions f_j has integral 0 on I. But the function f is not Riemann integrable on I. □

The last two examples show that something more than pointwise convergence is needed in order for the integral to respect the limit process.

Theorem: *Let f_j be integrable functions on a nontrivial bounded interval $[a, b]$ and suppose that the functions f_j converge uniformly to the limit function f. Then f is integrable on $[a, b]$ and*

$$\lim_{j\to\infty} \int_a^b f_j(x)\,dx = \int_a^b f(x)\,dx\,.$$

Proof: Pick $\epsilon > 0$. Choose N so large that, if $j > N$, then $|f_j(x) - f(x)| < \epsilon/[2(b-a)]$ for all $x \in [a, b]$. Notice that, if $j, k > N$, then

$$\left| \int_a^b f_j(x)\,dx - \int_a^b f_k(x)\,dx \right| \leq \int_a^b |f_j(x) - f_k(x)|\;dx\,.$$

But $|f_j(x) - f_k(x)| \leq |f_j(x) - f(x)| + |f(x) - f_k(x)| < \epsilon/(b-a)$. Therefore the last displayed line does not exceed

$$\int_a^b \frac{\epsilon}{b-a}\,dx = \epsilon\,.$$

Thus the numbers $\int_a^b f_j(x)\,dx$ form a Cauchy sequence. Let the limit of this sequence be called A.

Notice that, if we let $k \to \infty$ in the inequality

$$\left| \int_a^b f_j(x)\,dx - \int_a^b f_k(x)\,dx \right| \leq \epsilon\,,$$

then we obtain

$$\left| \int_a^b f_j(x)\,dx - A \right| \leq \epsilon$$

for all $j \geq N$. This estimate will be used below.

By hypothesis there is a $\delta > 0$ such that, if $\mathcal{P} = \{p_1, \ldots, p_k\}$ is a partition of $[a, b]$ with $m(\mathcal{P}) < \delta$, then

$$\left| \mathcal{R}(f_N, \mathcal{P}) - \int_a^b f_N(x)\, dx \right| < \epsilon\,.$$

But then, for such a partition, we have

$$\begin{aligned} |\mathcal{R}(f, \mathcal{P}) - A| \quad \leq \quad & \left| \mathcal{R}(f, \mathcal{P}) - \mathcal{R}(f_N, \mathcal{P}) \right| + \left| \mathcal{R}(f_N, \mathcal{P}) - \int_a^b f_N(x)\, dx \right| \\ & + \left| \int_a^b f_N(x)\, dx - A \right|. \end{aligned}$$

We have already noted that, by the choice of N, the third term on the right is smaller than ϵ. The second term is smaller than ϵ by the way that we chose the partition $\mathcal{P}$. It remains to examine the first term. Now

$$\begin{aligned}
\left|\mathcal{R}(f,\mathcal{P}) - \mathcal{R}(f_N,\mathcal{P})\right| &= \left|\sum_{j=1}^{k} f(s_j)\Delta_j - \sum_{j=1}^{k} f_N(s_j)\Delta_j\right| \\
&\leq \sum_{j=1}^{k}\left|f(s_j) - f_N(s_j)\right|\Delta_j \\
&< \sum_{j=1}^{k}\frac{\epsilon}{2(b-a)}\Delta_j \\
&= \frac{\epsilon}{2(b-a)}\sum_{j=1}^{k}\Delta_j \\
&= \frac{\epsilon}{2}.
\end{aligned}$$

Therefore $|\mathcal{R}(f,\mathcal{P}) - A| < 3\epsilon$ when $m(\mathcal{P}) < \delta$.

This shows that the function f is integrable on $[a, b]$ and has integral with value A. □

We have succeeded in answering questions **(1)** and **(2)** that were raised at the beginning of the section. In the next lecture we will answer questions **(3)**, **(4)**, and **(5)**.

Example: For each positive integer j, define

$$f_j(x) = \begin{cases} 0 & \text{if} \quad x \le j \\ x - j & \text{if} \quad j < x \le j+1 \\ (j+2) - x & \text{if} \quad j+1 < x \le j+2 \\ 0 & \text{if} \quad j+2 < x\,. \end{cases}$$

Then

$$\int f_j(x)\,dx = 1$$

for each j. But

$$\lim_{j\to\infty} f_j(x) = 0$$

for every x. So we see that

$$1 = \lim_{j\to\infty} \int f_j(x)\,dx \neq \int \lim_{j\to\infty} f_j(x)\,dx = \int 0\,dx = 0\,. \qquad \square$$