Math 4111
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Figure: This is your instructor.

Advanced Results on Integration Theory

We now turn to establishing the existence of certain Riemann–Stieltjes integrals.

Theorem: *Let f be continuous on $[a, b]$ and assume that α is monotonically increasing. Then*

$$\int_a^b f \, d\alpha$$

exists.

Proof: We may assume that α is nonconstant; otherwise there is nothing to prove.

Pick $\epsilon > 0$. By the uniform continuity of f we may choose a $\delta > 0$ such that if $|s - t| < \delta$ then $|f(s) - f(t)| < \epsilon/(\alpha(b) - \alpha(a))$. Let $\mathcal{P}$ be any partition of $[a, b]$ that has mesh smaller than δ. Then

$$\begin{aligned}
|\mathcal{U}(f, \mathcal{P}, \alpha) - \mathcal{L}(f, \mathcal{P}, \alpha)| &= \left| \sum_j M_j \Delta\alpha_j - \sum_j m_j \Delta\alpha_j \right| \\
&= \sum_j |M_j - m_j| \, \Delta\alpha_j \\
&< \sum_j \frac{\epsilon}{\alpha(b) - \alpha(a)} \Delta\alpha_j \\
&= \frac{\epsilon}{\alpha(b) - \alpha(a)} \cdot \sum_j \Delta\alpha_j \\
&= \epsilon \, .
\end{aligned}$$

Here, of course, we have used the monotonicity of α to observe that the last sum collapses to $\alpha(b) - \alpha(a)$. By Riemann's lemma, the proof is complete. $\square$

Notice how simple Riemann's lemma is to use. You may find it instructive to compare the proofs of this section with the rather difficult proofs in the second lecture on integration. What we are learning is that a good definition (and accompanying lemma(s)) can, in the end, make everything much clearer and simpler. Now we establish a companion result to the first one.

Theorem: *If α is an increasing and continuous function on the interval $[a, b]$ and if f is monotonic on $[a, b]$ then $\int_a^b f \, d\alpha$ exists.*

Proof: We may assume that $\alpha(b) > \alpha(a)$ and that f is monotone *increasing*. Let $L = \alpha(b) - \alpha(a)$ and $M = f(b) - f(a)$. Pick $\epsilon > 0$. Choose a positive integer k so that

$$\frac{L \cdot M}{k} < \epsilon \, .$$

Let $p_0 = a$ and choose p_1 to be the first point to the right of p_0 such that $\alpha(p_1) - \alpha(p_0) = L/k$ (this is possible, by the Intermediate Value Theorem, since α is continuous). Continuing, choose p_j to be the first point to the right of p_{j-1} such that $\alpha(p_j) - \alpha(p_{j-1}) = L/k$. This process will terminate after k steps and we will have $p_k = b$. Then $\mathcal{P} = \{p_0, p_1, \ldots, p_k\}$ is a partition of $[a, b]$.

Next observe that, for each j, the value M_j of $\sup f$ on I_j is $f(p_j)$ since f is increasing. Similarly the value m_j of $\inf f$ on I_j is $f(p_{j-1})$. We find therefore that

$$\begin{aligned}
\mathcal{U}(f,\mathcal{P},\alpha) - \mathcal{L}(f,\mathcal{P},\alpha) &= \sum_{j=1}^{k} M_j \Delta\alpha_j - \sum_{j=1}^{k} m_j \Delta\alpha_j \\
&= \sum_{j=1}^{k} \left((M_j - m_j)\frac{L}{k} \right) \\
&= \frac{L}{k} \sum_{j=1}^{k} (f(x_j) - f(x_{j-1})) \\
&= \frac{L \cdot M}{k} \\
&< \epsilon.
\end{aligned}$$

Therefore the inequality in Riemann's lemma is satisfied and the integral exists. □

One of the useful features of Riemann–Stieltjes integration is that it puts integration by parts into a very natural setting. We begin with a lemma.

Lemma: *Let f be continuous on an interval $[a, b]$ and let g be monotone increasing and continuous on that interval. If G is an antiderivative for g then*

$$\int_a^b f(x)g(x)\,dx = \int_a^b f\,dG\,.$$

Proof: Apply the Mean Value Theorem to the Riemann sums for the integral on the right. □

Theorem: [Integration by Parts] *Suppose that both f and g are continuous, increasing functions on the interval $[a, b]$. Let F be an antiderivative for f on $[a, b]$ and G an antiderivative for g on $[a, b]$. Then we have*

$$\int_a^b F\, dG = [F(b) \cdot G(b) - F(a) \cdot G(a)] - \int_a^b G\, dF\,.$$

Proof: Notice that, by the preceding lemma, both integrals exist. Set $P(x) = F(x) \cdot G(x)$. Then P has a continuous derivative on the interval $[a, b]$. Thus the Fundamental Theorem applies and we may write

$$P(b) - P(a) = \int_a^b P'(x)\,dx = [F(b) \cdot G(b) - F(a) \cdot G(a)]\ .$$

Now, writing out P' explicitly, using Leibnitz's Rule for the derivative of a product, we obtain

$$\int_a^b F(x)g(x)\,dx = [F(b)G(b) - F(a)G(a)] - \int_a^b G(x)f(x)\,dx\,.$$

But the lemma allows us to rewrite this equation as

$$\int_a^b F\,dG = [F(b)G(b) - F(a)G(a)] - \int_a^b G(x)\,dF\,. \qquad \square$$

Remark: The integration by parts formula can also be proved by applying *summation* by parts to the Riemann sums for the integral

$$\int_a^b F\, dG\,.$$

This method is explored in the exercises.

We have already observed that the Riemann–Stieltjes integral

$$\int_a^b f \, d\alpha$$

is linear in f; that is,

$$\int_a^b (f+g) \, d\alpha = \int_a^b f \, d\alpha + \int_a^b g \, d\alpha$$

and

$$\int_a^b c \cdot f \, d\alpha = c \cdot \int_a^b f \, d\alpha$$

when both f and g are Riemann–Stieltjes integrable with respect to α and for any constant c. We also would expect, from the very way that the integral is constructed, that it would be linear in the α entry.

But we have not even defined the Riemann–Stieltjes integral for nonincreasing α. And what of a function α that is the difference of two increasing functions? Such a function need not be monotone. Is it possible to identify which functions α can be decomposed as sums or differences of monotonic functions? It turns out that there is a satisfactory answer to these questions, and we should like to discuss these matters briefly.

Definition: If α is a monotonically *decreasing* function on $[a, b]$ and f is a function on $[a, b]$ then we define

$$\int_a^b f \, d\alpha = -\int_a^b f \, d(-\alpha)$$

when the right side exists.

The definition exploits the simple observation that if α is decreasing then $-\alpha$ is increasing; hence the preceding theory applies to the function $-\alpha$.

Next we have

Definition: Let α be a function on $[a, b]$ that can be expressed as

$$\alpha(x) = \alpha_1(x) - \alpha_2(x)\,,$$

where both α_1 and α_2 are increasing. Then, for any f on $[a, b]$, we define

$$\int_a^b f\,d\alpha = \int_a^b f\,d\alpha_1 - \int_a^b f\,d\alpha_2\,,$$

provided that both integrals on the right exist.

Now, by the very way that we have formulated our definitions, $\int_a^b f\,d\alpha$ is linear in both the f entry and the α entry. But the definitions are not satisfactory unless we can identify those α that can actually occur in the last definition. This leads us to a new class of functions.

Definition: Let f be a function on the interval $[a, b]$. For $x \in [a, b]$ we define

$$Vf(x) = \sup \sum_{j=1}^{k} |f(p_j) - f(p_{j-1})| \ ,$$

where the supremum is taken over all partitions $\mathcal{P} = \{p_0, p_1, \ldots, p_k\}$ of the interval $[a, x]$.

If $Vf \equiv Vf(b) < \infty$, then the function f is said to be of *bounded variation* on the interval $[a, b]$. In this circumstance the quantity $Vf(b)$ is called the *total variation* of f on $[a, b]$.

A function of bounded variation has the property that its graph does not have unbounded total oscillation.

Example: Define $f(x) = \sin x$, with domain the interval $[0, 2\pi]$. Let us calculate Vf. Let $\mathcal{P}$ be a partition of $[0, 2\pi]$. Since adding points to the partition only makes the sum

$$\sum_{j=1}^{k} |f(p_j) - f(p_{j-1})|$$

larger (by the triangle inequality), we may as well suppose that

$$\mathcal{P} = \{p_0, p_1, p_2, \ldots, p_k\}$$

contains the points $\pi/2, 3\pi/2$.

Assume that $p_{\ell_1} = \pi/2$ and $p_{\ell_2} = 3\pi/2$. Then

$$\begin{aligned}\sum_{j=1}^{k} |f(p_j) - f(p_{j-1})| &= \sum_{j=1}^{\ell_1} |f(p_j) - f(p_{j-1})| \\ &\quad + \sum_{j=\ell_1+1}^{\ell_2} |f(p_j) - f(p_{j-1})| \\ &\quad + \sum_{j=\ell_2+1}^{k} |f(p_j) - f(p_{j-1})| .\end{aligned}$$

However, f is increasing on the interval $[0, \pi/2] = [0, p_{\ell_1}]$.

Therefore the first sum is just

$$\sum_{j=1}^{\ell_1} f(p_j) - f(p_{j-1}) = f(p_{\ell_1}) - f(p_0) = f(\pi/2) - f(0) = 1\,.$$

Similarly, f is monotone on the intervals $[\pi/2, 3\pi/2] = [p_{\ell_1}, p_{\ell_2}]$ and $[3\pi/2, 2\pi] = [p_{\ell_2}, p_k]$. Thus the second and third sums equal $f(p_{\ell_1}) - f(p_{\ell_2}) = 2$ and $f(p_k) - f(p_{\ell_2}) = 1$ respectively. It follows that

$$Vf = Vf(2\pi) = 1 + 2 + 1 = 4\,.$$

Of course $Vf(x)$ for any $x \in [0, 2\pi]$ can be computed by similar means (see the exercises). □

Example: In general, if f is a continuously differentiable function on an interval $[a, b]$ then

$$Vf(x) = \int_a^x |f'(t)|\, dt\,.$$

This assertion will be explored in the exercises. □

Example: The function $f(x) = \cos x$ on the interval $[0, 2\pi]$ is of bounded variation. And in fact

$$Vf = 4$$

because the function goes from 1 down to 0 and then from 0 down to -1 and finally from -1 up to 0 and finally from 0 up to 1.

Alternatively, one can obtain the same answer by calculating the integral

$$Vf = \int_0^{2\pi} |f'(x)|\, dx = \int_0^{2\pi} |\sin x|\, dx\,. \qquad \square$$

Lemma: *Let f be a function of bounded variation on the interval $[a, b]$. Then the function Vf is increasing on $[a, b]$.*

Proof: Let $s < t$ be elements of $[a, b]$. Let $\mathcal{P} = \{p_0, p_1, \ldots, p_k\}$ be a partition of $[a, s]$. Then $\widetilde{\mathcal{P}} = \{p_0, p_1, \ldots, p_k, t\}$ is a partition of $[a, t]$ and

$$\sum_{j=1}^{k} |f(p_j) - f(p_{j-1})|$$

$$\begin{aligned} &\leq \sum_{j=1}^{k} |f(p_j) - f(p_{j-1})| + |f(t) - f(p_k)| \\ &\leq Vf(t). \end{aligned}$$

Taking the supremum on the left over all partitions $\mathcal{P}$ of $[a, s]$ yields that

$$Vf(s) \leq Vf(t).$$

□

Lemma: *Let f be a function of bounded variation on the interval $[a, b]$. Then the function $Vf - f$ is increasing on the interval $[a, b]$.*

Proof: Let $s < t$ be elements of $[a, b]$. Pick $\epsilon > 0$. By the definition of Vf we may choose a partition $\mathcal{P} = \{p_0, p_1, \ldots, p_k\}$ of the interval $[a, s]$ such that

$$Vf(s) - \epsilon < \sum_{j=1}^{k} |f(p_j) - f(p_{j-1})| .$$

But then $\widetilde{\mathcal{P}} = \{p_0, p_1, \ldots, p_k, t\}$ is a partition of $[a, t]$ and we have that

$$\sum_{j=1}^{k} |f(p_j) - f(p_{j-1})| + |f(t) - f(s)| \leq Vf(t) .$$

Using the inequality at the beginning of the proof, we may conclude that

$$Vf(s)-\epsilon+f(t)-f(s) < \sum_{j=1}^{k} |f(p_j)-f(p_{j-1})|+|f(t)-f(s)| \leq Vf(t)\,.$$

We conclude that

$$Vf(s) - f(s) < Vf(t) - f(t) + \epsilon\,.$$

Since the inequality holds for every $\epsilon > 0$, we see that the function $Vf - f$ is increasing. $\square$

Now we may combine the last two lemmas to obtain our main result:

Proposition: *If a function f is of bounded variation on $[a, b]$, then f may be written as the difference of two increasing functions. Conversely, the difference of two increasing functions is a function of bounded variation.*

Proof: *If f is of bounded variation write $f = Vf - (Vf - f) \equiv f_1 - f_2$. By the lemmas, both f_1 and f_2 are increasing.*

For the converse, assume that $f = f_1 - f_2$ with f_1, f_2 increasing. Then it is easy to see that

$$Vf(b) \leq |f_1(b) - f_1(a)| + |f_2(b) - f_2(a)| \ .$$

Thus f is of bounded variation. $\square$

Now the main point of this discussion is the following theorem:

Theorem: *If f is a continuous function on $[a, b]$ and if α is of bounded variation on $[a, b]$ then the integral*

$$\int_a^b f \, d\alpha$$

exists and is finite.

If g is of bounded variation on $[a, b]$ and if β is a continuous function of bounded variation on $[a, b]$ then the integral

$$\int_a^b g \, d\beta$$

exists and is finite.

Proof: Write the function(s) of bounded variation as the difference of increasing functions. Then apply the relevant theorems. □