Math 4111
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Figure: This is your instructor.

Another Look at the Integral

For many purposes, such as integration by parts, it is natural to formulate the integral in a more general context than we have considered in the first two lectures on integration. Our new formulation is called the *Riemann–Stieltjes integral* and is described below.

Fix an interval $[a, b]$ and a monotonically increasing function α on $[a, b]$. If $\mathcal{P} = \{p_0, p_1, \ldots, p_k\}$ is a partition of $[a, b]$, then let $\Delta\alpha_j = \alpha(p_j) - \alpha(p_{j-1})$. Let f be a bounded function on $[a, b]$ and define the *upper Riemann-Stieltjes sum* $\mathcal{U}$ of f with respect to α and the *lower Riemann-Stieltjes sum* $\mathcal{L}$ of f with respect to α as follows:

$$\mathcal{U}(f, \mathcal{P}, \alpha) = \sum_{j=1}^{k} M_j \Delta\alpha_j$$

and

$$\mathcal{L}(f, \mathcal{P}, \alpha) = \sum_{j=1}^{k} m_j \Delta\alpha_j \, .$$

Here the notation M_j denotes the supremum of f on the interval $I_j = [p_{j-1}, p_j]$ and m_j denotes the infimum of f on I_j.

In the special case $\alpha(x) = x$ the Riemann-Stieltjes sums discussed here have a form similar to the Riemann sums considered in the first two lectures. Moreover,

$$\mathcal{L}(f, \mathcal{P}, \alpha) \leq \mathcal{U}(f, \mathcal{P}, \alpha)\,.$$

We define

$$I^*(f) = \inf \mathcal{U}(f, \mathcal{P}, \alpha)$$

and

$$I_*(f) = \sup \mathcal{L}(f, \mathcal{P}, \alpha)\,.$$

Here the supremum and infimum are taken with respect to all partitions of the interval $[a, b]$. These are, respectively, the *upper* and *lower integrals* of f with respect to α on $[a, b]$.

By definition it is always true that, for any partition $\mathcal{P}$,

$$\mathcal{L}(f, \mathcal{P}, \alpha) \leq I_*(f) \leq I^*(f) \leq \mathcal{U}(f, \mathcal{P}, \alpha).$$

It is natural to declare the integral to exist when the upper and lower integrals agree:

Definition: Let α be an increasing function on the interval $[a, b]$ and let f be a bounded function on $[a, b]$. We say that the *Riemann–Stieltjes integral of f with respect to α* exists if

$$I^*(f) = I_*(f).$$

When the integral exists we denote it by

$$\int_a^b f \, d\alpha.$$

Notice that the definition of Riemann–Stieltjes integral is different from the definition of Riemann integral that we used in the preceding sections. It turns out that, when $\alpha(x) = x$, the two definitions are equivalent (this assertion is explored in the exercises). In the present generality it is easier to deal with upper and lower integrals in order to determine the existence of integrals.

Definition: Let $\mathcal{P}$ and $\mathcal{Q}$ be partitions of the interval $[a, b]$. If each point of $\mathcal{P}$ is also an element of $\mathcal{Q}$ then we call $\mathcal{Q}$ a *refinement* of $\mathcal{P}$.

Notice that the refinement $\mathcal{Q}$ is obtained by adding points to $\mathcal{P}$. The mesh of $\mathcal{Q}$ will be less than or equal to that of $\mathcal{P}$. The following lemma enables us to deal effectively with our new language.

Lemma: *Let $\mathcal{P}$ be a partition of the interval $[a, b]$ and f a function on $[a, b]$. Fix an increasing function α on $[a, b]$. If $\mathcal{Q}$ is a refinement of $\mathcal{P}$ then*

$$\mathcal{U}(f, \mathcal{Q}, \alpha) \leq \mathcal{U}(f, \mathcal{P}, \alpha)$$

and

$$\mathcal{L}(f, \mathcal{Q}, \alpha) \geq \mathcal{L}(f, \mathcal{P}, \alpha)\,.$$

Proof: Since $\mathcal{Q}$ is a refinement of $\mathcal{P}$ it holds that any interval I_ℓ arising from $\mathcal{Q}$ is contained in some interval $J_{j(\ell)}$ arising from $\mathcal{P}$. Let M_{I_ℓ} be the supremum of f on I_ℓ and $M_{J_{j(\ell)}}$ the supremum of f on the interval $J_{j(\ell)}$. Then $M_{I_\ell} \leq M_{J_{j(\ell)}}$. We conclude that

$$\mathcal{U}(f, \mathcal{Q}, \alpha) = \sum_\ell M_{I_\ell} \Delta\alpha_\ell \leq \sum_\ell M_{J_{j(\ell)}} \Delta\alpha_\ell \, .$$

We rewrite the right-hand side as

$$\sum_j M_{J_j} \left(\sum_{I_\ell \subseteq J_j} \Delta\alpha_\ell \right) \, .$$

However, because α is monotone, the inner sum simply equals $\alpha(p_j) - \alpha(p_{j-1}) = \Delta\alpha_j$. Thus the last expression is equal to $\mathcal{U}(f, \mathcal{P}, \alpha)$, as desired. In conclusion, $\mathcal{U}(f, \mathcal{Q}, \alpha) \leq \mathcal{U}(f, \mathcal{P}, \alpha)$.

A similar argument applies to the lower sums. $\square$

Example: Let $[a, b] = [0, 10]$ and let f be a continuous function on this interval. Let $\alpha(x)$ be the *greatest integer function*.[1] That is, $\alpha(x)$ is the greatest integer that does not exceed x. So, for example, $\alpha(0.5) = 0, \alpha(2) = 2$, and $\alpha(-3/2) = -2$. Then α is an increasing function on $[0, 10]$. Let f be any continuous function on $[0, 10]$. We shall determine whether

$$\int_0^{10} f \, d\alpha$$

exists and, if it does, calculate its value.

[1] In many texts the greatest integer in x is denoted by $[x]$. We do not use that notation here because it could get confused with our notation for a closed interval.

Let $\mathcal{P} = \{p_0, p_1, p_2, \ldots, p_k\}$ be a partition with mesh smaller than 1 of $[0, 10]$. By the lemma, it is to our advantage to assume that the mesh of $\mathcal{P}$ is smaller than 1. Observe that $\Delta\alpha_j$ equals the number of integers that lie in the interval I_j—that is, either 0 or 1. Let $I_{j_0}, I_{j_1}, \ldots I_{j_{10}}$ be, in sequence, the intervals from the partition which do in fact contain each distinct integer (the first of these contains 0, the second contains 1, and so on up to 10).

Then

$$\mathcal{U}(f,\mathcal{P},\alpha)=\sum_{\ell=0}^{10}M_{j_\ell}\Delta\alpha_{j_\ell}=\sum_{\ell=1}^{10}M_{j_\ell}$$

and

$$\mathcal{L}(f,\mathcal{P},\alpha)=\sum_{\ell=0}^{10}m_{j_\ell}\Delta\alpha_{j_\ell}=\sum_{\ell=1}^{10}m_{j_\ell}$$

because any term in these sums corresponding to an interval not containing an integer must have $\Delta\alpha_j=0$. Notice that $\Delta\alpha_{j_0}=0$ since $\alpha(0)=\alpha(p_0)=0$.

Let $\epsilon > 0$. Since f is uniformly continuous on $[0, 10]$, we may choose a $\delta > 0$ such that $|s - t| < \delta$ implies that $|f(s) - f(t)| < \epsilon/20$. If $m(\mathcal{P}) < \delta$ then it follows that $|f(\ell) - M_{j_\ell}| < \epsilon/20$ and $|f(\ell) - m_{j_\ell}| < \epsilon/20$ for $\ell = 0, 1, \ldots 10$. Therefore

$$\mathcal{U}(f, \mathcal{P}, \alpha) < \sum_{\ell=1}^{10} \left(f(\ell) + \frac{\epsilon}{20} \right)$$

and

$$\mathcal{L}(f, \mathcal{P}, \alpha) > \sum_{\ell=1}^{10} \left(f(\ell) - \frac{\epsilon}{20} \right) .$$

Rearranging these inequalities leads to

$$\mathcal{U}(f,\mathcal{P},\alpha) < \left(\sum_{\ell=1}^{10} f(\ell)\right) + \frac{\epsilon}{2}$$

and

$$\mathcal{L}(f,\mathcal{P},\alpha) > \left(\sum_{\ell=1}^{10} f(\ell)\right) - \frac{\epsilon}{2}\,.$$

Thus, since $I_*(f)$ and $I^*(f)$ are trapped between $\mathcal{U}$ and $\mathcal{L}$, we conclude that

$$|I_*(f) - I^*(f)| < \epsilon\,.$$

We have seen that, if the partition is fine enough, then the upper and lower integrals of f with respect to α differ by at most ϵ. It follows that $\int_0^{10} f\,d\alpha$ exists. Moreover,

$$\left| I^*(f) - \sum_{\ell=1}^{10} f(\ell) \right| < \epsilon$$

and

$$\left| I_*(f) - \sum_{\ell=1}^{10} f(\ell) \right| < \epsilon\,.$$

We conclude that

$$\int_0^{10} f\,d\alpha = \sum_{\ell=1}^{10} f(\ell)\,. \qquad \square$$

The example demonstrates that the language of the Riemann–Stieltjes integral allows us to think of the integral as a generalization of the summation process. This is frequently useful, both philosophically and for practical reasons.

The next result, sometimes called Riemann's lemma, is crucial for proving the existence of Riemann–Stieltjes integrals.

Proposition: [Riemann's Lemma] *Let α be an increasing function on $[a, b]$ and f a bounded function on the interval. The Riemann–Stieltjes integral of f with respect to α exists if and only if, for every $\epsilon > 0$, there is a partition $\mathcal{P}$ such that*

$$|\mathcal{U}(f, \mathcal{P}, \alpha) - \mathcal{L}(f, \mathcal{P}, \alpha)| < \epsilon \,.$$

Proof: First assume that the inequality holds. Fix $\epsilon > 0$. Since $\mathcal{L} \leq I_* \leq I^* \leq \mathcal{U}$, the inequality implies that

$$|I^*(f) - I_*(f)| < \epsilon\,.$$

But this means that $\int_a^b f\,d\alpha$ exists.

Conversely, assume that the integral exists. Fix $\epsilon > 0$. Choose a partition $\mathcal{Q}_1$ such that

$$|\mathcal{U}(f, \mathcal{Q}_1, \alpha) - I^*(f)| < \epsilon/2\,.$$

Likewise choose a partition $\mathcal{Q}_2$ such that

$$|\mathcal{L}(f, \mathcal{Q}_2, \alpha) - I_*(f)| < \epsilon/2\,.$$

Since $I_*(f) = I^*(f)$ it follows that

$$|\mathcal{U}(f, \mathcal{Q}_1, \alpha) - \mathcal{L}(f, \mathcal{Q}_2, \alpha)| < \epsilon\,.$$

Let $\mathcal{P}$ be the common refinement of $\mathcal{Q}_1$ and $\mathcal{Q}_2$. Then we have, again by the lemma, that

$$\mathcal{L}(f, \mathcal{Q}_2, \alpha) \leq \mathcal{L}(f, \mathcal{P}, \alpha) \leq \int_a^b f\, d\alpha \leq \mathcal{U}(f, \mathcal{P}, \alpha) \leq \mathcal{U}(f, \mathcal{Q}_1, \alpha)\,.$$

But the expressions on the far left and on the far right of these inequalities differ by less than ϵ. Thus $\mathcal{P}$ satifies the inequality. $\square$

We note in passing that the basic properties of the Riemann integral noted in the second lecture on the integral hold without change for the Riemann–Stieltjes integral. The proofs are left as exercises for you (use Riemann's lemma!).