Math 4111
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Figure: This is your instructor.

Change of Variable and Related Ideas

Another fundamental operation in the theory of the integral is "change of variable" (sometimes called the "u-substitution" in calculus books). We next turn to a careful formulation and proof of this operation. First we need a lemma:

Lemma: *If f is a Riemann integrable function on $[a, b]$ and if ϕ is a continuous function on a compact interval that contains the range of f then $\phi \circ f$ is Riemann integrable.*

Proof: Let $\epsilon > 0$. Since ϕ is a continuous function on a compact set, it is uniformly continuous. Let $\delta > 0$ be selected such that **(i)** $\delta < \epsilon$ and **(ii)** if $|x - y| < \delta$ then $|\phi(x) - \phi(y)| < \epsilon$.

Now the hypothesis that f is Riemann integrable implies that there exists a $\widetilde{\delta} > 0$ such that if $\mathcal{P}$ and $\mathcal{P}'$ are partitions of $[a, b]$ and $m(\mathcal{P}), m(\mathcal{P}') < \widetilde{\delta}$ then, for the common refinement $\mathcal{Q}$ of $\mathcal{P}$ and $\mathcal{P}'$, it holds that

$$|\mathcal{R}(f, \mathcal{P}) - \mathcal{R}(f, \mathcal{Q})| < \delta^2 \quad \text{and} \quad \left|\mathcal{R}(f, \widetilde{\mathcal{P}}) - \mathcal{R}(f, \mathcal{Q})\right| < \delta^2 .$$

Fix such a $\mathcal{P}, \mathcal{P}'$ and $\mathcal{Q}$. Let J_ℓ be the intervals of $\mathcal{Q}$ and I_j the intervals of $\mathcal{P}$. Each J_ℓ is contained in some $I_{j(\ell)}$. We write

$$
\begin{aligned}
&\left|\mathcal{R}(\phi\circ f,\mathcal{P})-\mathcal{R}(\phi\circ f,\mathcal{Q})\right| \\
&\quad= \left|\sum_j \phi\circ f(t_j)\Delta_j - \sum_\ell \phi\circ f(s_\ell)\Delta_\ell\right| \\
&\quad= \left|\sum_j\sum_{J_\ell\subseteq I_j} \phi\circ f(t_j)\Delta_\ell - \sum_j\sum_{J_\ell\subseteq I_j} \phi\circ f(s_\ell)\Delta_\ell\right| \\
&\quad= \left|\sum_j\sum_{J_\ell\subseteq I_j} \Big[\phi\circ f(t_j)-\phi\circ f(s_\ell)\Big]\Delta_\ell\right| \\
&\quad\le \left|\sum_j\sum_{J_\ell\subseteq I_j,\ell\in G} \Big[\phi\circ f(t_j)-\phi\circ f(s_\ell)\Big]\Delta_\ell\right| \\
&\qquad+\left|\sum_j\sum_{J_\ell\subseteq I_j,\ell\in B} \Big[\phi\circ f(t_j)-\phi\circ f(s_\ell)\Big]\Delta_\ell\right|,
\end{aligned}
$$

where we put ℓ in G if $J_\ell \subseteq I_{j(\ell)}$ and

$0 \leq \left(\sup_{I_{j(\ell)}} f - \inf_{I_{j(\ell)}} f \right) < \delta$; otherwise we put ℓ into B.

Notice that

$$\begin{aligned}
\sum_{\ell \in B} \delta \Delta_\ell &\leq \sum_{\ell \in B} \left(\sup_{I_{j(\ell)}} f - \inf_{I_{j(\ell)}} f \right) \cdot \Delta_\ell \\
&= \sum_{j=1}^{k} \sum_{J_\ell \subseteq I_j} \left(\sup_{I_j} f - \inf_{I_j} f \right) \cdot \Delta_\ell \\
&= \sum_{j=1}^{k} \left(\sup_{I_j} f - \inf_{I_j} f \right) \Delta_j \\
&< \delta^2
\end{aligned}$$

by the choice of $\widetilde{\delta}$ (see the remark). Therefore

$$\sum_{\ell \in B} \Delta_\ell < \delta \, .$$

Let M be an upper bound for $|\phi|$. Then

$$\begin{aligned}\left|\sum_j \sum_{J_\ell \subseteq I_j, \ell \in B} \Big(\phi \circ f(t_j) - \phi \circ f(s_\ell)\Big)\Delta_\ell\right| &\leq \left|\sum_j \sum_{J_\ell \subseteq I_j, \ell \in B} \Big(2 \cdot M\Big)\Delta_\ell\right| \\ &\leq 2 \cdot \delta \cdot M \\ &< 2M\epsilon.\end{aligned}$$

Also

$$\left|\sum_j \sum_{J_\ell \subseteq I_j, \ell \in G} \left(\phi \circ f(t_j) - \phi \circ f(s_\ell)\right)\Delta_\ell\right| \leq \left|\sum_j \sum_{J_\ell \subseteq I_j, \ell \in G} \epsilon\Delta_\ell\right|$$

since, for $\ell \in G$, we know that $|f(\alpha) - f(\beta)| < \delta$ for any $\alpha, \beta \in I_{j(\ell)}$. However, the last line does not exceed $(b-a) \cdot \epsilon$.

Putting together our estimates, we find that

$$|\mathcal{R}(\phi \circ f, \mathcal{P}) - \mathcal{R}(\phi \circ f, \mathcal{Q})| < \epsilon \cdot (2M + (b - a)) .$$

By symmetry, an analogous inequality holds for $\mathcal{P}'$. By the relevant lemma, this is what we needed to prove. □

An easier result is that, if f is Riemann integrable on an interval $[a, b]$ and if $\mu : [\alpha, \beta] \to [a, b]$ is continuously differentiable, then $f \circ \mu$ is Riemann integrable (see the exercises).

Corollary: *If f and g are Riemann integrable on $[a, b]$, then so is the function $f \cdot g$.*

Proof: By an earlier theorem, $f + g$ is integrable. By the lemma, $(f + g)^2 = f^2 + 2f \cdot g + g^2$ is integrable. But the lemma also implies that f^2 and g^2 are integrable (here we use the function $\phi(x) = x^2$). It results, by subtraction, that $2 \cdot f \cdot g$ is integrable. Hence $f \cdot g$ is integrable. $\square$

Theorem: *Let f be an integrable function on an interval $[a, b]$ of positive length. Let ψ be a continuously differentiable function from another interval $[\alpha, \beta]$ of positive length into $[a, b]$. Assume that ψ is increasing, one-to-one, and onto. Then*

$$\int_a^b f(x)\,dx = \int_\alpha^\beta f(\psi(x)) \cdot \psi'(x)\,dx\,.$$

Proof: Since f is integrable, its absolute value is bounded by some number M. Fix $\epsilon > 0$. Since ψ' is continuous on the compact interval $[\alpha, \beta]$, it is uniformly continuous. Hence we may choose $\delta > 0$ so small that if $|s - t| < \delta$ then $|\psi'(s) - \psi'(t)| < \epsilon / (M \cdot (\beta - \alpha))$. If $\mathcal{P} = \{p_0, \ldots, p_k\}$ is any partition of $[a, b]$ then there is an associated partition $\widetilde{\mathcal{P}} = \{\psi^{-1}(p_0), \ldots, \psi^{-1}(p_k)\}$ of $[\alpha, \beta]$. For simplicity denote the points of $\widetilde{\mathcal{P}}$ by $\widetilde{p}_j$. Let us choose the partition $\mathcal{P}$ so fine that the mesh of $\widetilde{\mathcal{P}}$ is less than δ. If t_j are points of $I_j = [p_{j-1}, p_j]$ then there are corresponding points $s_j = \psi^{-1}(t_j)$ of $\widetilde{I}_j = [\widetilde{p}_{j-1}, \widetilde{p}_j]$.

Then we have

$$\begin{aligned}
\sum_{j=1}^{k} f(t_j)\Delta_j &= \sum_{j=1}^{k} f(t_j)(p_j - p_{j-1}) \\
&= \sum_{j=1}^{k} f(\psi(s_j))(\psi(\widetilde{p}_j) - \psi(\widetilde{p}_{j-1})) \\
&= \sum_{j=1}^{k} f(\psi(s_j))\psi'(u_j)(\widetilde{p}_j - \widetilde{p}_{j-1}),
\end{aligned}$$

where we have used the Mean Value Theorem in the last line to find each u_j. Our problem at this point is that $f \circ \psi$ and ψ' are evaluated at different points. So we must do some estimation to correct that problem.

The last displayed line equals

$$\sum_{j=1}^{k} f(\psi(s_j))\psi'(s_j)(\widetilde{p}_j - \widetilde{p}_{j-1}) + \sum_{j=1}^{k} f(\psi(s_j))\left(\psi'(u_j) - \psi'(s_j)\right)(\widetilde{p}_j - \widetilde{p}_{j-1}).$$

The first sum is a Riemann sum for $f(\psi(x)) \cdot \psi'(x)$ and the second sum is an error term. Since the points u_j and s_j are elements of the same interval $\widetilde{I}_j$ of length less than δ, we conclude that $|\psi'(u_j) - \psi'(s_j)| < \epsilon/(M \cdot |\beta - \alpha|)$. Thus the error term in absolute value does not exceed

$$\sum_{j=1}^{k} M \cdot \frac{\epsilon}{M \cdot |\beta - \alpha|} \cdot (\widetilde{p}_j - \widetilde{p}_{j-1}) = \frac{\epsilon}{\beta - \alpha} \sum_{j=0}^{k} (\widetilde{p}_j - \widetilde{p}_{j-1}) = \epsilon.$$

This shows that every Riemann sum for f on $[a, b]$ with sufficiently small mesh corresponds to a Riemann sum for $f(\psi(x)) \cdot \psi'(x)$ on $[\alpha, \beta]$ plus an error term of size less than ϵ. A similar argument shows that every Riemann sum for $f(\psi(x)) \cdot \psi'(x)$ on $[\alpha, \beta]$ with sufficiently small mesh corresponds to a Riemann sum for f on $[a, b]$ plus an error term of magnitude less than ϵ. The conclusion is then that the integral of f on $[a, b]$ (which exists by hypothesis) and the integral of $f(\psi(x)) \cdot \psi'(x)$ on $[\alpha, \beta]$ (which exists by the corollary to the lemma) agree. $\square$

Example: Let us analyze the integral

$$\int_0^1 \sin(x^3 + x) \cdot (3x^2 + 1)\, dx\,.$$

We let $f(t) = \sin t$ and $\psi(x) = x^3 + x$. Then we see that the integral has the form

$$\int_0^1 f \circ \psi(x) \cdot \psi'(x)\, dx\,.$$

Here $\psi : [0, 1] \to [0, 2]$.

By the theorem, this integral is equal to

$$\int_0^2 f(t)\, dt = \int_0^2 \sin t\, dt\,. \qquad \square$$

We conclude this lecture with the very important

Theorem: [Fundamental Theorem of Calculus]
Let f be an integrable function on the interval $[a, b]$. For $x \in [a, b]$ we define

$$F(x) = \int_a^x f(s)\, ds\,.$$

If f is continuous at $x \in (a, b)$ then

$$F'(x) = f(x)\,.$$

Proof: Fix $x \in (a, b)$. Let $\epsilon > 0$. Choose, by the continuity of f at x, a $\delta > 0$ such that $|s - x| < \delta$ implies $|f(s) - f(x)| < \epsilon$. We may assume that $\delta < \min\{x - a, b - x\}$. If $|t - x| < \delta$ then

$$\begin{aligned}
\left| \frac{F(t) - F(x)}{t - x} - f(x) \right| &= \left| \frac{\int_a^t f(s)\, ds - \int_a^x f(s)\, ds}{t - x} - f(x) \right| \\
&= \left| \frac{\int_x^t f(s)\, ds}{t - x} - \frac{\int_x^t f(x)\, ds}{t - x} \right| \\
&= \left| \frac{\int_x^t (f(s) - f(x))\ ds}{t - x} \right|.
\end{aligned}$$

Notice that we rewrote $f(x)$ as the integral with respect to a dummy variable s over an interval of length $|t-x|$ divided by $(t-x)$. Assume for the moment that $t > x$. Then the last line is dominated by

$$\begin{aligned} \frac{\int_x^t |f(s)-f(x)|\, ds}{t-x} &\leq \frac{\int_x^t \epsilon\, ds}{t-x} \\ &= \epsilon. \end{aligned}$$

A similar estimate holds when $t < x$ (simply reverse the limits of integration).

This shows that

$$\lim_{t \to x} \frac{F(t) - F(x)}{t - x}$$

exists and equals $f(x)$. Thus $F'(x)$ exists and equals $f(x)$. $\square$

In the exercises we shall consider how to use the theory of one-sided limits to make the conclusion of the Fundamental Theorem true on the entire interval $[a, b]$. We conclude with

Corollary: *If f is a continuous function on $[a, b]$ and if G is any continuously differentiable function on $[a, b]$ whose derivative equals f on (a, b) then*

$$\int_a^b f(x)\,dx = G(b) - G(a)\,.$$

Proof: Define F as in the theorem. Since F and G have the same derivative on (a, b), they differ by a constant. Then

$$\int_a^b f(x)\,dx = F(b) - F(a) = G(b) - G(a)$$

as desired. □

Example: Let

$$f(x) = \int_0^{x^2} \cos(e^t)\, dt\,.$$

What is the derivative of f?

It is not possible to actually evaluate the given integral, but we can still answer the question. Let $g(s) = \int_0^s \cos(e^t)\, dt$ and let $h(x) = x^2$. Then $f = g \circ h$. Therefore

$$f'(x) = g'(h(x)) \cdot h'(x)\,.$$

Now the Fundamental Theorem of Calculus tells us that

$$g'(s) = \cos(e^s)\,.$$

And obviously $h'(x) = 2x$.

In conclusion,

$$f'(x) = \cos(e^{x^2}) \cdot 2x\,. \qquad \square$$