Math 4111
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Steven G. Krantz



Figure: This is your instructor.

Properties of the Riemann Integral

We begin this section with a few elementary properties of the integral that reflect its linear nature.

Theorem: *Let $[a, b]$ be a nonempty, bounded interval, let f and g be Riemann integrable functions on the interval, and let α be a real number. Then $f \pm g$ and $\alpha \cdot f$ are integrable and we have*

(a) $\int_a^b f(x) \pm g(x)\, dx = \int_a^b f(x)\, dx \pm \int_a^b g(x)\, dx$;

(b) $\int_a^b \alpha \cdot f(x)\, dx = \alpha \cdot \int_a^b f(x)\, dx$.

Proof: For **(a)**, let

$$A = \int_a^b f(x)\,dx$$

and

$$B = \int_a^b g(x)\,dx\,.$$

Let $\epsilon > 0$. Choose a $\delta_1 > 0$ such that if $\mathcal{P}$ is a partition of $[a, b]$ with mesh less than δ_1 then

$$|\mathcal{R}(f, \mathcal{P}) - A| < \frac{\epsilon}{2}\,.$$

Similarly choose a $\delta_2 > 0$ such that if $\mathcal{P}$ is a partition of $[a, b]$ with mesh less than δ_2 then

$$|\mathcal{R}(g, \mathcal{P}) - B| < \frac{\epsilon}{2}\,.$$

Let $\delta = \min\{\delta_1, \delta_2\}$. If $\mathcal{P}'$ is any partition of $[a, b]$ with $m(\mathcal{P}') < \delta$ then

$$\begin{aligned}
\left|\mathcal{R}(f \pm g, \mathcal{P}') - (A \pm B)\right| &= \left|\mathcal{R}(f, \mathcal{P}') \pm \mathcal{R}(g, \mathcal{P}') - (A \pm B)\right| \\
&\leq \left|\mathcal{R}(f, \mathcal{P}') - A\right| + \left|\mathcal{R}(g, \mathcal{P}') - B\right| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon.
\end{aligned}$$

This means that the integral of $f \pm g$ exists and equals $A \pm B$, as we were required to prove.

The proof of **(b)** follows similar lines but is much easier and we leave it as an exercise for you. □

Theorem: *If c is a point of the interval $[a, b]$ and if f is Riemann integrable on both $[a, c]$ and $[c, b]$ then f is integrable on $[a, b]$ and*

$$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx\,.$$

Proof: Let us write

$$A = \int_a^c f(x)\,dx$$

and

$$B = \int_c^b f(x)\,dx\,.$$

Now pick $\epsilon > 0$. There is a $\delta_1 > 0$ such that if $\mathcal{P}$ is a partition of $[a, c]$ with mesh less than δ_1 then

$$|\mathcal{R}(f, \mathcal{P}) - A| < \frac{\epsilon}{3}\,.$$

Similarly, choose $\delta_2 > 0$ such that if $\mathcal{P}'$ is a partition of $[c, b]$ with mesh less than δ_2 then

$$\left|\mathcal{R}(f, \mathcal{P}') - B\right| < \frac{\epsilon}{3}.$$

Let M be an upper bound for $|f|$ (recall, from the remark at the end of the last lecture, that a Riemann integrable function must be bounded).

Set $\delta = \min\{\delta_1, \delta_2, \epsilon/(6M)\}$. Now let $\mathcal{V} = \{v_1, \ldots, v_k\}$ be any partition of $[a, b]$ with mesh less than δ. There is a last point v_n which is in $[a, c]$ and a first point v_{n+1} in $[c, b]$. Observe that $\mathcal{P} = \{v_0, \ldots, v_n, c\}$ is a partition of $[a, c]$ with mesh smaller than δ_1 and $\mathcal{P}' = \{c, v_{n+1}, \ldots, v_k\}$ is a partition of $[c, b]$ with mesh smaller than δ_2. Let us rename the elements of $\mathcal{P}$ as $\{p_0, \ldots, p_{n+1}\}$ and the elements of $\mathcal{P}'$ as $\{p_0^{'}, \cdots p_{k-n+1}^{'}\}$. Notice that $p_{n+1} = p_0^{'} = c$. For each j let s_j be a point chosen in the interval $I_j = [v_{j-1}, v_j]$ from the partition $\mathcal{V}$.

Then we have

$$\left|\mathcal{R}(f,\mathcal{V}) - \Big[A+B\Big]\right|$$

$$= \left|\left(\sum_{j=1}^{n} f(s_j)\Delta_j - A\right) + f(s_{n+1})\Delta_{n+1} + \left(\sum_{j=n+2}^{k} f(s_j)\Delta_j - B\right)\right|$$

$$= \left|\left(\sum_{j=1}^{n} f(s_j)\Delta_j + f(c)\cdot(c-v_n) - A\right)\right.$$

$$\left. + \left(f(c)\cdot(v_{n+1}-c) + \sum_{j=n+2}^{k} f(s_j)\Delta_j - B\right)\right.$$

$$
\begin{aligned}
&\quad + \Big(f(s_{n+1}) - f(c)\Big) \cdot (c - v_n) + \Big(f(s_{n+1}) - f(c)\Big) \cdot (v_{n+1} - c)\Big| \\
&\leq \left|\left(\sum_{j=1}^{n} f(s_j)\Delta_j + f(c) \cdot (c - v_n) - A\right)\right| \\
&\quad + \left|\left(f(c) \cdot (v_{n+1} - c) + \sum_{j=n+2}^{k} f(s_j)\Delta_j - B\right)\right| \\
&\quad + \Big|(f(s_{n+1}) - f(c)) \cdot (v_{n+1} - v_n)\Big|
\end{aligned}
$$

$$
\begin{aligned}
&= \left|\mathcal{R}(f,\mathcal{P}) - A\right| + \left|\mathcal{R}(f,\mathcal{P}') - B\right| \\
&\quad + \left|\left(f(s_{n+1}) - f(c)\right)\cdot(v_{n+1} - v_n)\right| \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} + 2M\cdot\delta \\
&\leq \epsilon
\end{aligned}
$$

by the choice of δ.

This shows that f is integrable on the entire interval $[a, b]$ and the value of the integral is

$$A + B = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx\,. \qquad \square$$

Remark: The last proof illustrates why it is useful to be able to choose the $s_j \in I_j$ arbitrarily.

Example: If we adopt the convention that

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

(which is consistent with the way that the integral was defined in the first place), then the last theorem is true even when c is not an element of $[a, b]$. For instance, suppose that $c < a < b$. Then, by the theorem,

$$\int_c^a f(x)\,dx + \int_a^b f(x)\,dx = \int_c^b f(x)\,dx\,.$$

But this may be rearranged to read

$$\int_a^b f(x)\,dx = -\int_c^a f(x)\,dx + \int_c^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx\,.$$

One of the basic tools of analysis is to perform estimates. Thus we require certain fundamental inequalities about integrals. These are recorded in the next theorem.

Theorem: *Let f and g be integrable functions on a nonempty interval $[a, b]$. Then*

(i) $\left| \int_a^b f(x)\,dx \right| \leq \int_a^b |f(x)|\,dx;$

(ii) *If $f(x) \leq g(x)$ for all $x \in [a, b]$ then*
$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx.$

Proof: If $\mathcal{P}$ is any partition of $[a, b]$ then

$$|\mathcal{R}(f, \mathcal{P})| \leq \mathcal{R}(|f|, \mathcal{P}) .$$

The first assertion follows.

Next, for part **(ii)**,

$$\mathcal{R}(f, \mathcal{P}) \leq \mathcal{R}(g, \mathcal{P}) .$$

This inequality implies the second assertion. □

Example: We may estimate the integral

$$\int_0^1 \sin^3 x \, dx$$

as follows. We apply the last theorem, part (i), to see that

$$\left| \int_0^1 \sin^3 x \, dx \right| \leq \int_0^1 |\sin^3 x| \, dx \, .$$

Now we apply part (ii) of the theorem to determine that

$$\int_0^1 |\sin^3 x| \, dx \leq \int_0^1 1 \, dx = 1 \, . \qquad \square$$