Math 4111
December 2, 2020 Lecture

Steven G. Krantz



Figure: This is your instructor.

Remark: We mention now a useful fact that will be formalized in later sections. Suppose that f is Riemann integrable on $[a, b]$ with the value of the integral being ℓ. Let $\epsilon > 0$. Then, as stated in the definition (with $\epsilon/2$ replacing ϵ), there is a $\delta > 0$ such that, if $\mathcal{Q}$ is a partition of $[a, b]$ of mesh smaller than δ, then $|\mathcal{R}(f, \mathcal{Q}) - \ell| < \epsilon/2$. It follows that, if $\mathcal{P}$ and $\mathcal{P}'$ are partitions of $[a, b]$ of mesh smaller than δ, then

$$\left|\mathcal{R}(f, \mathcal{P}) - \mathcal{R}(f, \mathcal{P}')\right| \leq |\mathcal{R}(f, \mathcal{P}) - \ell| + |\ell - \mathcal{R}(f, \mathcal{P}')| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon\,.$$

This is like a Cauchy condition on partitions.

Note, however, that we may choose $\mathcal{P}'$ to equal the partition $\mathcal{P}$. Also we may for each j choose the point s_j, where f is evaluated for the Riemann sum over $\mathcal{P}$, to be a point where f very nearly assumes its supremum on I_j. Likewise we may for each j choose the point s'_j, where f is evaluated for the Riemann sum over $\mathcal{P}'$, to be a point where f very nearly assumes its infimum on I_j. It easily follows that, when the mesh of $\mathcal{P}$ is less than δ, then

$$\sum_j \left(\sup_{I_j} f - \inf_{I_j} f \right) \Delta_j \leq \epsilon \,. \tag{7.7.1}$$

This consequence of integrability will prove useful to us in some of the discussions in this and the next lecture. In the exercises we shall consider in detail the assertion that integrability implies (7.7.1) and the converse as well.

Definition If $\mathcal{P}, \mathcal{P}'$ are partitions of $[a, b]$ then their *common refinement* is the union of all the points of $\mathcal{P}$ and $\mathcal{P}'$. See Figure 7.3.

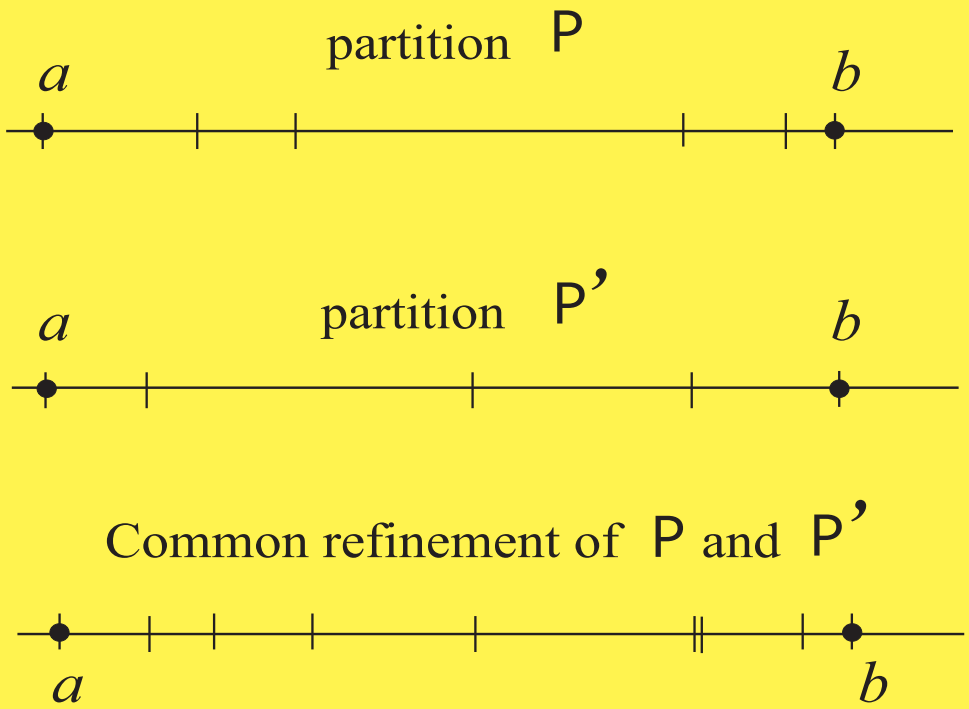


Figure: The common refinement.

We record now a technical lemma that will be used in several of the proofs that follow:

Lemma: Let f be a function with domain the closed interval $[a, b]$. The Riemann integral

$$\int_a^b f(x)\,dx$$

exists if and only if, for every $\epsilon > 0$, there is a $\delta > 0$ such that, if $\mathcal{P}$ and $\mathcal{P}'$ are partitions of $[a, b]$ with $m(\mathcal{P}) < \delta$ and $m(\mathcal{P}') < \delta$, then their common refinement $\mathcal{Q}$ has the property that

$$|\mathcal{R}(f, \mathcal{P}) - \mathcal{R}(f, \mathcal{Q})| < \epsilon$$

and (7.9.1)

$$|\mathcal{R}(f, \mathcal{P}') - \mathcal{R}(f, \mathcal{Q})| < \epsilon\,.$$

Proof: If f is Riemann integrable then the assertion of the lemma follows immediately from the definition of the integral.

For the converse note that (7.9.1) certainly implies that, if $\epsilon > 0$, then there is a $\delta > 0$ such that, if $\mathcal{P}$ and $\mathcal{P}'$ are partitions of $[a, b]$ with $m(\mathcal{P}) < \delta$ and $m(\mathcal{P}') < \delta$, then

$$|\mathcal{R}(f, \mathcal{P}) - \mathcal{R}(f, \mathcal{P}')| < \epsilon \tag{7.9.2}$$

(just use the triangle inequality).

Now, for each $\epsilon_j = 2^{-j}, j = 1, 2, \ldots,$ we can choose a $\delta_j > 0$ as in (7.9.2). Let S_j be the *closure* of the set

$$\{\mathcal{R}(f, \mathcal{P}) : m(\mathcal{P}) < \delta_j\}.$$

By the choice of δ_j, the set S_j is contained in a closed interval of length not greater than $2\epsilon_j$.

On the one hand,

$$\bigcap_j S_j$$

must be nonempty since it is the decreasing intersection of compact sets. On the other hand, the length estimate implies that the intersection must be contained in a closed interval of length 0—that is, the intersection is a point. That point is then the limit of the Riemann sums, that is, it is the value of the Riemann integral. □

The most important, and perhaps the simplest, fact about the Riemann integral is that a large class of familiar functions is Riemann integrable.

Theorem: *Let f be a continuous function on a nontrivial closed, bounded interval $I = [a, b]$. Then f is Riemann integrable on $[a, b]$.*

Proof: We use the lemma. Given $\epsilon > 0$, choose (by the uniform continuity of f on I) a $\delta > 0$ such that, whenever $|s - t| < \delta$ then

$$|f(s) - f(t)| < \frac{\epsilon}{b - a}\,. \tag{7.10.1}$$

Let $\mathcal{P}$ and $\mathcal{P}'$ be any two partitions of $[a, b]$ of mesh smaller than δ. Let $\mathcal{Q}$ be the common refinement of $\mathcal{P}$ and $\mathcal{P}'$.

Now we let I_j denote the intervals arising in the partition $\mathcal{P}$ (and having length Δ_j) and $\widetilde{I}_\ell$ the intervals arising in the partition $\mathcal{Q}$ (and having length $\widetilde{\Delta}_\ell$). Since the partition $\mathcal{Q}$ contains every point of $\mathcal{P}$, plus some additional points as well, every $\widetilde{I}_\ell$ is contained in some I_j.

Fix j and consider the expression

$$\left| f(s_j)\Delta_j - \sum_{\widetilde{I}_\ell \subseteq I_j} f(t_\ell)\widetilde{\Delta}_\ell \right| . \tag{7.10.2}$$

We write

$$\Delta_j = \sum_{\widetilde{I}_\ell \subseteq I_j} \widetilde{\Delta}_\ell \, .$$

This equality enables us to rearrange (7.10.2) as

$$\begin{aligned} & \left| f(s_j) \cdot \sum_{\widetilde{I}_\ell \subseteq I_j} \widetilde{\Delta}_\ell - \sum_{\widetilde{I}_\ell \subseteq I_j} f(t_\ell) \widetilde{\Delta}_\ell \right| \\ = & \left| \sum_{\widetilde{I}_\ell \subseteq I_j} [f(s_j) - f(t_\ell)] \widetilde{\Delta}_\ell \right| \\ \leq & \sum_{\widetilde{I}_\ell \subseteq I_j} |f(s_j) - f(t_\ell)| \widetilde{\Delta}_\ell. \end{aligned}$$

But each of the points t_ℓ is in the interval I_j, as is s_j. So they differ by less than δ. Therefore, by (7.10.1), the last expression is less than

$$\begin{aligned}\sum_{\widetilde{I}_\ell \subseteq I_j} \frac{\epsilon}{b-a}\widetilde{\Delta}_\ell &= \frac{\epsilon}{b-a}\sum_{\widetilde{I}_\ell \subseteq I_j} \widetilde{\Delta}_\ell \\ &= \frac{\epsilon}{b-a}\cdot \Delta_j.\end{aligned}$$

Now we conclude the argument by writing

$$\begin{aligned}
|\mathcal{R}(f,\mathcal{P}) - \mathcal{R}(f,\mathcal{Q})| &= \left|\sum_j f(s_j)\Delta_j - \sum_\ell f(t_\ell)\widetilde{\Delta}_\ell\right| \\
&\leq \sum_j \left|f(s_j)\Delta_j - \sum_{\widetilde{I}_\ell \subseteq I_j} f(t_\ell)\widetilde{\Delta}_\ell\right| \\
&< \sum_j \frac{\epsilon}{b-a}\cdot\Delta_j \\
&= \frac{\epsilon}{b-a}\cdot\sum_j \Delta_j \\
&= \frac{\epsilon}{b-a}\cdot(b-a) \\
&= \epsilon.
\end{aligned}$$

The estimate for $|\mathcal{R}(f,\mathcal{P}') - \mathcal{R}(f,\mathcal{Q})|$ is identical and we omit it. The result now follows from the lemma. □

In the exercises we will ask you to extend the theorem to the case of functions f on $[a, b]$ that are bounded and have finitely many, or even countably many, discontinuities.

We conclude this section by noting an important fact about Riemann integrable functions. A Riemann integrable function on an interval $[a, b]$ *must be bounded*. If it were not, then one could choose the points s_j in the construction of $\mathcal{R}(f, \mathcal{P})$ so that $f(s_j)$ is arbitrarily large, and the Riemann sums would become arbitrarily large, hence cannot converge. You will be asked in the exercises to work out the details of this assertion.