Math 4111
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Figure: This is your instructor.

Partitions and the Concept of Integral

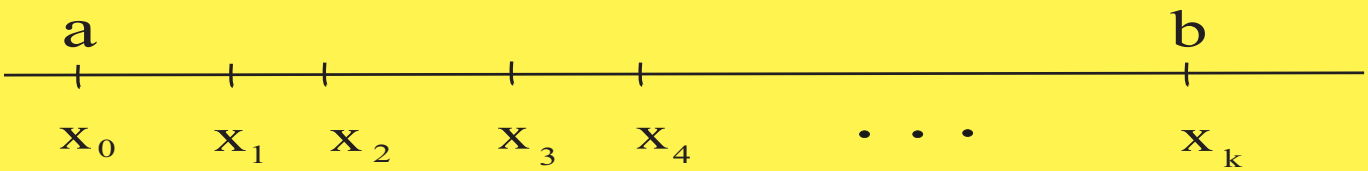
We learn in calculus that it is often useful to think of an integral as representing area. However, this is but one of many important applications of integration theory. The integral is a generalization of the summation process. That is the point of view that we shall take in the present lecture.

Definition: Let $[a, b]$ be a closed interval in $\mathbb{R}$. A finite, ordered set of points $\mathcal{P} = \{x_0, x_1, x_2, \ldots, x_{k-1}, x_k\}$ such that

$$a = x_0 \leq x_1 \leq x_2 \leq \cdots \leq x_{k-1} \leq x_k = b$$

is called a *partition* of $[a, b]$. Refer to Figure 7.1.

If $\mathcal{P}$ is a partition of $[a, b]$, then we let I_j denote the interval $[x_{j-1}, x_j]$, $j = 1, 2, \ldots, k$. The symbol Δ_j denotes the *length* of I_j. The *mesh* of $\mathcal{P}$, denoted by $m(\mathcal{P})$, is defined to be $\max_j \Delta_j$.


a
b
x 0
x 1
x 2
x 3
x 4
. . .
x k

The points of a partition need not be equally spaced, nor must they be distinct from each other.

Example The set $\mathcal{P} = \{0, 1, 1, 9/8, 2, 5, 21/4, 23/4, 6\}$ is a partition of the interval $[0, 6]$ with mesh 3 (because $I_5 = [2, 5]$, with length 3, is the longest interval in the partition). See Figure 7.2. □


9/8
21/4
6
0
1
2
5
23/4

Definition Let $[a, b]$ be an interval and let f be a function with domain $[a, b]$. If $\mathcal{P} = \{x_0, x_1, x_2, \ldots, x_{k-1}, x_k\}$ is a partition of $[a, b]$ and if, for each j, s_j is an element of I_j, then the corresponding *Riemann sum* is defined to be

$$\mathcal{R}(f, \mathcal{P}) = \sum_{j=1}^{k} f(s_j)\Delta_j \, .$$

Example Let $f(x) = x^2 - x$ and $[a, b] = [1, 4]$. Define the partition $\mathcal{P} = \{1, 3/2, 2, 7/3, 4\}$ of this interval. Then a Riemann sum for this f and $\mathcal{P}$ is

$$\begin{aligned}\mathcal{R}(f, \mathcal{P}) &= \left(1^2 - 1\right) \cdot \frac{1}{2} + \left((7/4)^2 - (7/4)\right) \cdot \frac{1}{2} \\ &\quad + \left((7/3)^2 - (7/3)\right) \cdot \frac{1}{3} + \left(3^2 - 3\right) \cdot \frac{5}{3} \\ &= \frac{10103}{864}.\end{aligned}$$

□

Notice that we have complete latitude in choosing each point s_j from the corresponding interval I_j. While at first confusing, we will find this freedom to be a powerful tool when proving results about the integral.

The first main step in the theory of the Riemann integral is to determine a method for "calculating the limit of the Riemann sums" of a function as the mesh of the partitions tends to zero. There are in fact several methods for doing so. We have chosen the simplest one.

Definition Let $[a, b]$ be an interval and f a function with domain $[a, b]$. We say that *the Riemann sums of f tend to a limit ℓ as $m(\mathcal{P})$ tends to 0* if, for any $\epsilon > 0$, there is a $\delta > 0$ such that, if $\mathcal{P}$ is any partition of $[a, b]$ with $m(\mathcal{P}) < \delta$, then $|\mathcal{R}(f, \mathcal{P}) - \ell| < \epsilon$ for every choice of $s_j \in I_j$.

It will turn out to be critical for the success of this definition that we require that *every* partition of mesh smaller than δ satisfy the conclusion of the definition. The theory does not work effectively if for every $\epsilon > 0$ there is a $\delta > 0$ and *some* partition $\mathcal{P}$ of mesh less than δ which satisfies the conclusion of the definition.

Definition A function f on a closed interval $[a, b]$ is said to be *Riemann integrable* on $[a, b]$ if the Riemann sums of $\mathcal{R}(f, \mathcal{P})$ tend to a finite limit ℓ as $m(\mathcal{P})$ tends to zero.

The value ℓ of the limit, when it exists, is called the *Riemann integral* of f over $[a, b]$ and is denoted by

$$\int_a^b f(x)\,dx\,.$$