Math 4111
November 20, 2020 Lecture

Steven G. Krantz



Figure: This is your instructor.

More on the Theory of Differentiation

l'Hôpital's Rule (actually due to his teacher J. Bernoulli (1667–1748)) is a useful device for calculating limits, and a nice application of the Cauchy Mean Value Theorem. Here we present a special case of the theorem.

Theorem: *Suppose that f and g are differentiable functions on an open interval I and that $p \in I$. If $\lim_{x\to p} f(x) = \lim_{x\to p} g(x) = 0$ and if*

$$\lim_{x\to p} \frac{f'(x)}{g'(x)}$$

exists and equals a real number ℓ then

$$\lim_{x\to p} \frac{f(x)}{g(x)} = \ell\,.$$

Proof: Fix a real number $a > \ell$. By the formula there is a number $q > p$ such that, if $p < x < q$, then

$$\frac{f'(x)}{g'(x)} < a\,.$$

But now, if $p < s < t < q$, then

$$\frac{f(t) - f(s)}{g(t) - g(s)} = \frac{f'(x)}{g'(x)}$$

for some $s < x < t$ (by Cauchy's Mean Value Theorem). It follows then from the equation above that

$$\frac{f(t) - f(s)}{g(t) - g(s)} < a\,.$$

Now let $s \to p$ and invoke the hypothesis about the zero limit of f and g at p to conclude that

$$\frac{f(t)}{g(t)} \leq a$$

when $p < t < q$. Since a is an arbitrary number to the right of ℓ we conclude that

$$\limsup_{t \to p^+} \frac{f(t)}{g(t)} \leq \ell \, .$$

Similar arguments show that

$$\liminf_{t \to p^+} \frac{f(t)}{g(t)} \geq \ell \, ;$$

$$\limsup_{t \to p^-} \frac{f(t)}{g(t)} \leq \ell \, ;$$

$$\liminf_{t \to p^-} \frac{f(t)}{g(t)} \geq \ell \, .$$

We conclude that the desired limit exists and equals ℓ.

A closely related result, with a similar proof, is this:

Theorem: *Suppose that f and g are differentiable functions on an open interval I and that $p \in I$. If $\lim_{x\to p} f(x) = \lim_{x\to p} g(x) = \pm\infty$ and if*

$$\lim_{x\to p} \frac{f'(x)}{g'(x)}$$

exists and equals a real number ℓ then

$$\lim_{x\to p} \frac{f(x)}{g(x)} = \ell\,.$$

Example: Let

$$f(x) = \left|\ln|x|\right|^{(x^2)}.$$

We wish to determine $\lim_{x\to 0} f(x)$. To do so, we define

$$F(x) = \ln f(x) = x^2 \ln\left|\ln|x|\right| = \frac{\ln|\ln|x||}{1/x^2}.$$

Notice that both the numerator and the denominator tend to $\pm\infty$ as $x \to 0$. So the hypotheses of l'Hôpital's rule are satisfied and the limit is

$$\lim_{x\to 0} \frac{\ln|\ln|x||}{1/x^2} = \lim_{x\to 0} \frac{(\ln|\ln|x|)'}{(1/x^2)'} = \lim_{x\to 0} \frac{1/[x\ln|x|]}{-2/x^3} = \lim_{x\to 0} \frac{-x^2}{2\ln|x|} = 0.$$

Since $\lim_{x\to 0} F(x) = 0$ we may calculate that the original limit has value $\lim_{x\to 0} f(x) = 1$. □

Proposition: *Let f be an invertible function on an interval (a, b) with nonzero derivative at a point $x \in (a, b)$. Let $X = f(x)$. Then $\left(f^{-1}\right)'(X)$ exists and equals $1/f'(x)$.*

Proof: Observe that, for $T \neq X$,

$$\frac{f^{-1}(T) - f^{-1}(X)}{T - X} = \frac{1}{\frac{f(t)-f(x)}{t-x}},$$

where $T = f(t)$. Since $f'(x) \neq 0$, the difference quotients for f in the denominator are bounded from zero hence the limit of the formula exists. This proves that f^{-1} is differentiable at X and that the derivative at that point equals $1/f'(x)$. □

Example: We know that the function $f(x) = x^k$, k a positive integer, is one-to-one and differentiable on the interval $(0, 1)$. Moreover the derivative $k \cdot x^{k-1}$ never vanishes on that interval. Therefore the proposition applies and we find for $X \in (0, 1) = f((0, 1))$ that

$$(f^{-1})'(X) = \frac{1}{f'(x)} = \frac{1}{f'(X^{1/k})}$$

$$= \frac{1}{k \cdot X^{1-1/k}} = \frac{1}{k} \cdot X^{1/k-1}.$$

In other words,

$$\left(X^{1/k}\right)' = \frac{1}{k} X^{1/k-1}. \qquad \square$$

We conclude this lecture by saying a few words about higher derivatives. If f is a differentiable function on an open interval I then we may ask whether the function f' is differentiable. If it is, then we denote its derivative by

$$f'' \quad \text{or} \quad f^{(2)} \quad \text{or} \quad \frac{d^2}{dx^2} f \quad \text{or} \quad \frac{d^2 f}{dx^2},$$

and call it the second derivative of f. Likewise the derivative of the $(k-1)$th derivative, if it exists, is called the kth derivative and is denoted

$$f''^{\cdots\prime} \quad \text{or} \quad f^{(k)} \quad \text{or} \quad \frac{d^k}{dx^k} f \quad \text{or} \quad \frac{d^k f}{dx^k}.$$

Observe that we cannot even consider whether $f^{(k)}$ exists at a point unless $f^{(k-1)}$ exists in a *neighborhood* of that point.

If f is k times differentiable on an open interval I and if each of the derivatives $f^{(1)}, f^{(2)}, \ldots, f^{(k)}$ is continuous on I then we say that the function f is *k times continuously differentiable* on I. We write $f \in C^k(I)$. Obviously there is some redundancy in this definition since the continuity of $f^{(j-1)}$ follows from the existence of $f^{(j)}$. Thus only the continuity of the last derivative $f^{(k)}$ need be checked. Continuously differentiable functions are useful tools in analysis. We denote the class of k times continuously differentiable functions on I by $C^k(I)$.

Example: For $k = 1, 2, \ldots$ the function

$$f_k(x) = \begin{cases} x^{k+1} & \text{if} \quad x \geq 0 \\ -x^{k+1} & \text{if} \quad x < 0 \end{cases}$$

will be k times continuously differentiable on $\mathbb{R}$ but will fail to be $k+1$ times differentiable at $x = 0$. More dramatically, an analysis similar to the one we used on the Weierstrass nowhere differentiable function shows that the function

$$g_k(x) = \sum_{j=1}^{\infty} \frac{3^j}{4^{j+jk}} \sin(4^j x)$$

is k times continuously differentiable on $\mathbb{R}$ but will not be $k+1$ times differentiable at any point (this function, with $k = 0$, was Weierstrass's original example). □

A more refined notion of smoothness/continuity of functions is that of Hölder continuity or Lipschitz continuity (see an earlier lecture). If f is a function on an open interval I and if $0 < \alpha \leq 1$ then we say that f satisfies a *Lipschitz condition* of order α on I if there is a constant M such that for all $s, t \in I$ we have

$$|f(s) - f(t)| \leq M \cdot |s - t|^{\alpha}.$$

Such a function is said to be of class $\mathrm{Lip}_{\alpha}(I)$. Clearly a function of class Lip_{α} is uniformly continuous on I. For, if $\epsilon > 0$, then we may take $\delta = (\epsilon/M)^{1/\alpha}$: it follows that, for $|s - t| < \delta$, we have

$$|f(s) - f(t)| \leq M \cdot |s - t|^{\alpha} < M \cdot \epsilon/M = \epsilon.$$

Interestingly, when $\alpha > 1$ the class Lip_α contains only constant functions. For in this instance the inequality

$$|f(s) - f(t)| \leq M \cdot |s - t|^\alpha$$

leads to

$$\left| \frac{f(s) - f(t)}{s - t} \right| \leq M \cdot |s - t|^{\alpha - 1}.$$

Because $\alpha - 1 > 0$, letting $s \to t$ yields that $f'(t)$ exists for every $t \in I$ and equals 0. It follows from a corollary in the last lecture that f is constant on I.

Instead of trying to extend the definition of $\mathrm{Lip}_\alpha(I)$ to $\alpha > 1$ it is customary to define classes of functions $C^{k,\alpha}$, for $k = 0, 1, \ldots$ and $0 < \alpha \leq 1$, by the condition that f be of class C^k on I and that $f^{(k)}$ be an element of $\mathrm{Lip}_\alpha(I)$. We leave it as an exercise for you to verify that $C^{k,\alpha} \subseteq C^{\ell,\beta}$ if either $k > \ell$ or both $k = \ell$ and $\alpha \geq \beta$.

In more advanced studies in analysis, it is appropriate to replace $\mathrm{Lip}_1(I)$, and more generally $C^{k,1}$, with another space (invented by Antoni Zygmund, 1900–1992) defined in a more subtle fashion using second differences as in an example at the end of the last lecture. These matters exceed the scope of this book, but we shall make a few remarks about them in the exercises.