Math 4111
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Figure: This is your instructor.

More on the Mean Value Theorem

Proof: If f is a constant function then any point ξ in the interval will do. So assume that f is nonconstant.

An earlier theorem guarantees that f will have both a maximum and a minimum in $[a, b]$. If one of these occurs in (a, b) then Fermat's proposition guarantees that f' will vanish at that point and we are done. If both occur at the endpoints then all the values of f lie between 0 and 0. In other words f is constant, contradicting our assumption. □

Example: Of course the point ξ in Rolle's theorem need not be unique. If $f(x) = x^3 - x^2 - 2x$ on the interval $[-1, 2]$ then $f(-1) = f(2) = 0$ and $f'(x) = 3x^2 - 2x - 2$ vanishes at *two* points (namely, $(2 + \sqrt{28})/6$ and $(2 - \sqrt{28})/6$) of the interval $(-1, 2)$. Refer to the next figure. □

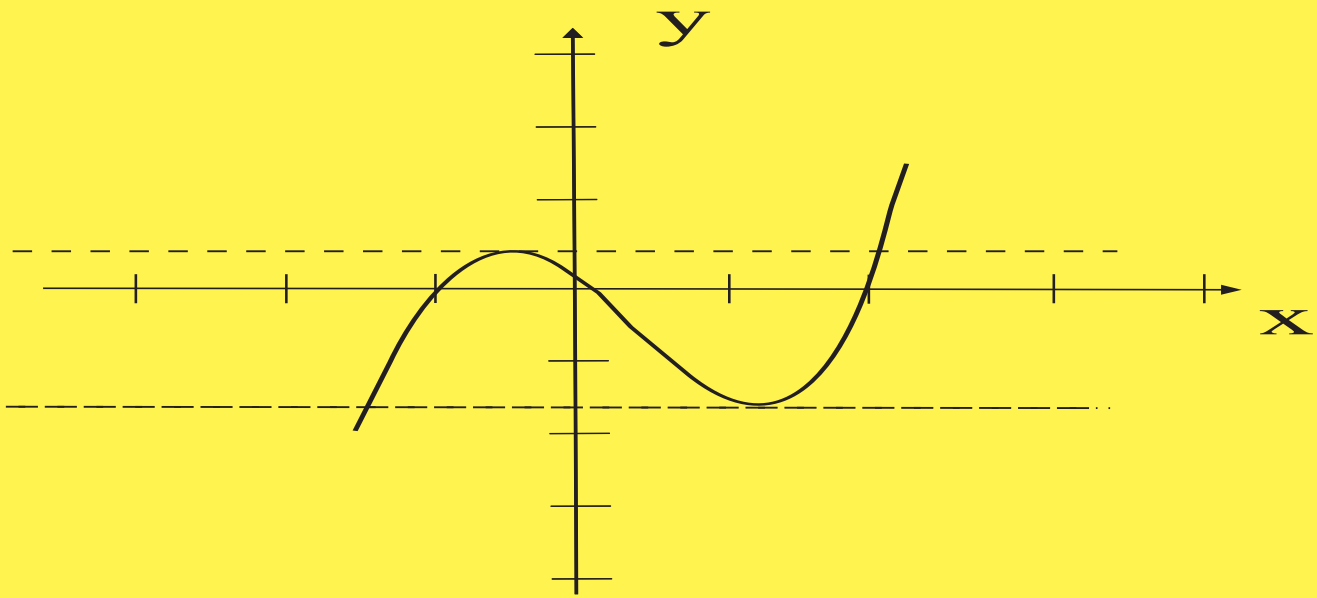


Figure: An example of Rolle's theorem.

If you rotate the graph of a function satisfying the hypotheses of Rolle's theorem, the result suggests that, for any continuous function f on an interval $[a, b]$, differentiable on (a, b), we should be able to relate the slope of the chord connecting $(a, f(a))$ and $(b, f(b))$ with the value of f' at some interior point. That is the content of the standard Mean Value Theorem:

Theorem: [The Mean Value Theorem] *Let f be a continuous function on the closed interval $[a, b]$ that is differentiable on (a, b). There exists a point $\xi \in (a, b)$ such that*

$$\frac{f(b) - f(a)}{b - a} = f'(\xi)\,.$$

See the next figure.

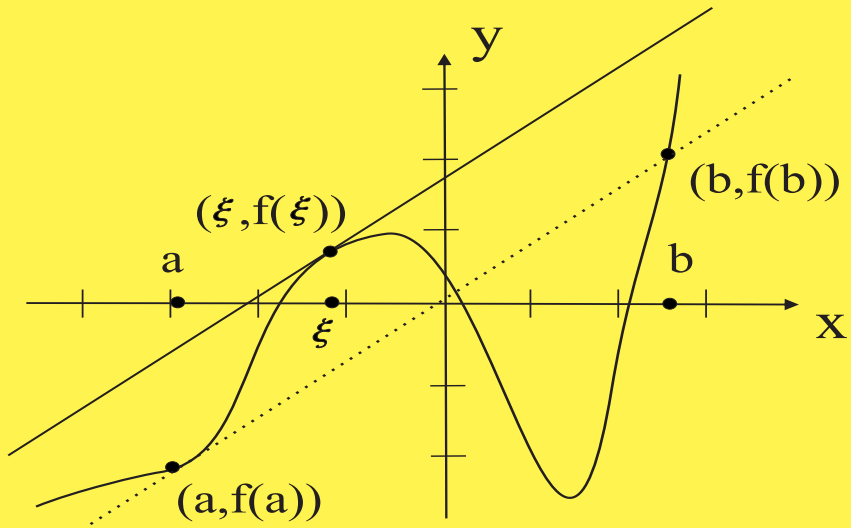


Figure: The Mean Value Theorem.

Proof: Our scheme is to implement the remarks preceding the theorem: we "rotate" the picture to reduce to the case of Rolle's theorem. More precisely, define

$$g(x) = f(x) - \left[f(a) + \frac{f(b) - f(a)}{b - a} \cdot (x - a) \right] \quad \text{if} \quad x \in [a, b] .$$

By direct verification, g is continuous on $[a, b]$ and differentiable on (a, b) (after all, g is obtained from f by elementary arithmetic operations). Also $g(a) = g(b) = 0$. Thus we may apply Rolle's theorem to g and we find that there is a $\xi \in (a, b)$ such that $g'(\xi) = 0$. Remembering that x is the variable, we differentiate the formula for g to find that

$$\begin{aligned} 0 = g'(\xi) &= \left[f'(x) - \frac{f(b) - f(a)}{b - a} \right] \Bigg|_{x=\xi} \\ &= \left[f'(\xi) - \frac{f(b) - f(a)}{b - a} \right]. \end{aligned}$$

As a result,

$$f'(\xi) = \frac{f(b) - f(a)}{b - a}.$$

□

Corollary: *If f is a differentiable function on the open interval I and if $f'(x) = 0$ for all $x \in I$ then f is a constant function.*

Proof: *If s and t are any two elements of I then the theorem tells us that*

$$f(s) - f(t) = f'(\xi) \cdot (s - t)$$

for some ξ between s and t. But, by hypothesis, $f'(\xi) = 0$. We conclude that $f(s) = f(t)$. But, since s and t were chosen arbitrarily, we must conclude that f is constant. □

Corollary: *If f is differentiable on an open interval I and $f'(x) \geq 0$ for all $x \in I$, then f is increasing on I; that is, if $s < t$ are elements of I, then $f(s) \leq f(t)$.*

If f is differentiable on an open interval I and $f'(x) \leq 0$ for all $x \in I$, then f is decreasing on I; that is, if $s < t$ are elements of I, then $f(s) \geq f(t)$.

Proof: Similar to the preceding corollary. □

Example: Let us verify that, if f is a differentiable function on $\mathbb{R}$, and if $|f'(x)| \leq 1$ for all x, then $|f(s) - f(t)| \leq |s - t|$ for all real s and t.

In fact, for $s \neq t$ there is a ξ between s and t such that

$$\frac{f(s) - f(t)}{s - t} = f'(\xi)\,.$$

But $|f'(\xi)| \leq 1$ by hypothesis hence

$$\left|\frac{f(s) - f(t)}{s - t}\right| = |f'(\xi)| \leq 1$$

or

$$|f(s) - f(t)| \leq |s - t|\,. \qquad \square$$

Example Let us verify that

$$\lim_{x\to+\infty}\left(\sqrt{x+5}-\sqrt{x}\right)=0\,.$$

Here the limit operation means that, for any $\epsilon>0$, there is an $N>0$ such that $x>N$ implies that the expression in parentheses has absolute value less than ϵ.

Define $f(x)=\sqrt{x}$ for $x>0$. Then the expression in parentheses is just $f(x+5)-f(x)$. By the Mean Value Theorem this equals

$$f'(\xi)\cdot 5$$

for some $x<\xi<x+5$. But this last expression is

$$\frac{1}{2}\cdot\xi^{-1/2}\cdot 5\,.$$

By the bounds on ξ, this is

$$\leq \frac{5}{2} x^{-1/2} .$$

Clearly, as $x \to +\infty$, this expression tends to zero.

A powerful tool in analysis is a generalization of the usual Mean Value Theorem that is due to A. L. Cauchy (1789–1857):

Theorem: [Cauchy's Mean Value Theorem] *Let f and g be continuous functions on the interval $[a, b]$ which are both differentiable on the interval (a, b), $a < b$. Assume that $g' \neq 0$ on the interval and that $g(a) \neq g(b)$. Then there is a point $\xi \in (a, b)$ such that*

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(\xi)}{g'(\xi)}.$$

Proof: Apply the usual Mean Value Theorem to the function

$$h(x) = g(x) \cdot \{f(b) - f(a)\} - f(x) \cdot \{g(b) - g(a)\}. \qquad \square$$

Clearly the usual Mean Value Theorem is obtained from Cauchy's by taking $g(x)$ to be the function x. We conclude this section by illustrating a typical application of the result.

Example: Let f be a differentiable function on an interval I such that f' is differentiable at a point $x \in I$. Then

$$\lim_{h \to 0^+} \frac{f(x+h) + f(x-h) - 2f(x)}{h^2} = (f')'(x) \equiv f''(x)\,.$$

To see this, fix x and define $\mathcal{F}(h) = f(x+h) + f(x-h) - 2f(x)$ and $\mathcal{G}(h) = h^2$. Then

$$\frac{f(x+h) + f(x-h) - 2f(x)}{h^2} = \frac{\mathcal{F}(h) - \mathcal{F}(0)}{\mathcal{G}(h) - \mathcal{G}(0)}\,.$$

According to Cauchy's Mean Value Theorem, there is a ξ between 0 and h such that the last line equals

$$\frac{\mathcal{F}'(\xi)}{\mathcal{G}'(\xi)}.$$

Writing this last expression out gives

$$\begin{aligned}\frac{f'(x+\xi)-f'(x-\xi)}{2\xi} &= \frac{1}{2}\cdot\frac{f'(x+\xi)-f'(x)}{\xi} \\ &\quad + \frac{1}{2}\cdot\frac{f'(x-\xi)-f'(x)}{-\xi},\end{aligned}$$

and the last line tends as $h \to 0$, by the definition of the derivative, to the quantity $(f')'(x)$. □

It is a fact that the standard proof of l'Hôpital's Rule (Guillaume François Antoine de l'Hôpital, Marquis de St.-Mesme, 1661–1704) is obtained by way of Cauchy's Mean Value Theorem. This line of reasoning is explored in the next lecture.