# Math 4111
# November 20, 2020 Lecture

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Figure: This is your instructor.

# The Mean Value Theorem and Applications

We begin this lecture with a definition about local maxima and minima of functions.

**Definition:** Let $f$ be a function with domain $(a, b)$. A point $C \in (a, b)$ is called a *local maximum* for $f$ (we also say that $f$ has a local maximum at $C$) if there is a $\delta > 0$ such that $f(t) \leq f(C)$ for all $t \in (C - \delta, C + \delta)$. A point $c \in (a, b)$ is called a *local minimum* for $f$ (we also say that $f$ has a local minimum at $c$) if there is a $\delta > 0$ such that $f(t) \geq f(c)$ for all $t \in (c - \delta, c + \delta)$. See the figure.

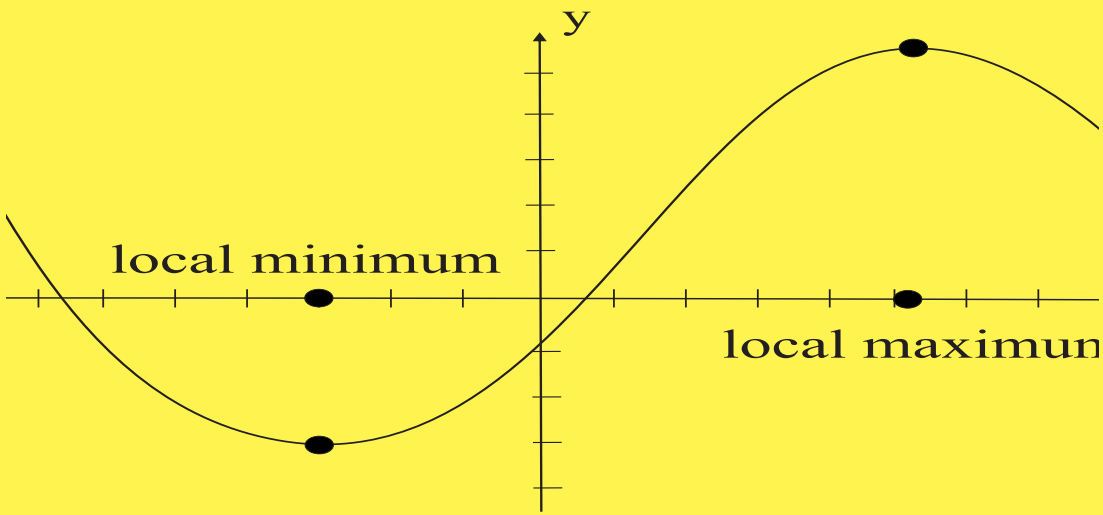



Figure: Some extrema.

Local minima (plural of minimum) and local maxima (plural of maximum) are referred to collectively as *local extrema*.

**Proposition:** [Fermat] *If $f$ is a function with domain $(a, b)$, if $f$ has a local extremum at $x \in (a, b)$, and if $f$ is differentiable at $x$, then $f'(x) = 0$.*

**Proof:** Suppose that $f$ has a local minimum at $x$. Then there is a $\delta > 0$ such that if $x - \delta < t < x$ then $f(t) \geq f(x)$. Then

$$\frac{f(t) - f(x)}{t - x} \leq 0\,.$$

Letting $t \to x$, it follows that $f'(x) \leq 0$. Similarly, if $x < t < x + \delta$ for suitable $\delta$, then

$$\frac{f(t) - f(x)}{t - x} \geq 0\,.$$

It follows that $f'(x) \geq 0$. We must conclude that $f'(x) = 0$.

A similar argument applies if $f$ has a local maximum at $x$. The proof is therefore complete. $\square$

**Example:** Consider the function $f(x) = \sin x$ on the interval $[\pi/3, 11\pi/3]$. Surely $f'(x) = \cos x$, and we see that $f'$ vanishes at the points $\pi/2$, $3\pi/2$, $5\pi/2$, $7\pi/2$ of the interval. By Fermat's theorem, these are candidates to be local maxima or minima of $f$. And, indeed, $\pi/2$, $5\pi/2$ are local maxima and $3\pi/2$, $7\pi/2$ are local minima. □

Before going on to mean value theorems, we provide a striking application of the proposition:

**Theorem:** [Darboux's Theorem] *Let $f$ be a differentiable function on an open interval $I$. Pick points $s < t$ in $I$ and suppose that $f'(s) < \rho < f'(t)$. Then there is a point $u$ between $s$ and $t$ such that $f'(u) = \rho$.*

**Proof:** Consider the function $g(x) = f(x) - \rho x$. Then $g'(s) < 0$ and $g'(t) > 0$. Assume for simplicity that $s < t$. The sign of the derivative at $s$ shows that $g(\widehat{s}) < g(s)$ for $\widehat{s}$ greater than $s$ and near $s$. The sign of the derivative at $t$ implies that $g(\widehat{t}) < g(t)$ for $\widehat{t}$ less than $t$ and near $t$. Thus the minimum of the continuous function $g$ on the compact interval $[s, t]$ must occur at some point $u$ in the interior $(s, t)$. The preceding proposition guarantees that $g'(u) = 0$, or $f'(u) = \rho$ as claimed. □

**Example:** If $f'$ were a continuous function then the theorem would just be a special instance of the Intermediate Value Property of continuous functions (see an earlier corollary). But derivatives need not be continuous, as the example

$$f(x) = \begin{cases} x^{3/2} \cdot \sin(1/x) & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad x = 0 \end{cases}$$

illustrates. Check for yourself that $f'(0)$ exists and vanishes but $\lim_{x \to 0} f'(x)$ does not exist. This example illustrates the significance of the theorem. □

Since the theorem says that $f'$ will always satisfy the Intermediate Value Property (even when it is not continuous), its discontinuities cannot be of the first kind. In other words:

**Proposition:** *If $f$ is a differentiable function on an open interval $I$ then the discontinuities of $f'$ are all of the second kind.*

Next we turn to the simplest form of the Mean Value Theorem.

**Theorem:** [Rolle's Theorem] *Let $f$ be a continuous function on the closed interval $[a, b]$ which is differentiable on $(a, b)$. If $f(a) = f(b) = 0$ then there is a point $\xi \in (a, b)$ such that $f'(\xi) = 0$. See the next figure.*

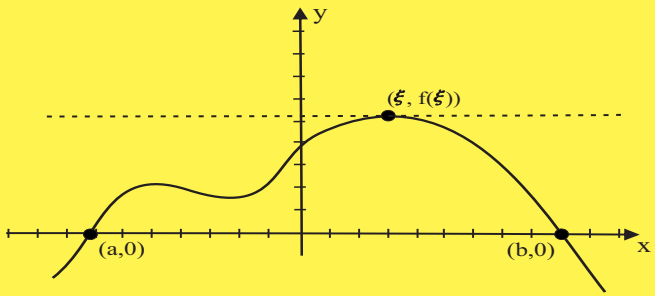



Figure: Rolle's theorem.