Math 4111
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Figure: This is your instructor.

Let f be a function with domain an open interval I. If $x \in I$ then the quantity

$$\frac{f(t) - f(x)}{t - x}$$

measures the slope of the chord of the graph of f that connects the points $(x, f(x))$ and $(t, f(t))$. See the first figure. If we let $t \to x$ then the limit of the quantity represented by this "Newton quotient" should represent the slope of the graph *at the point* x. These considerations motivate the definition of the derivative:

Definition

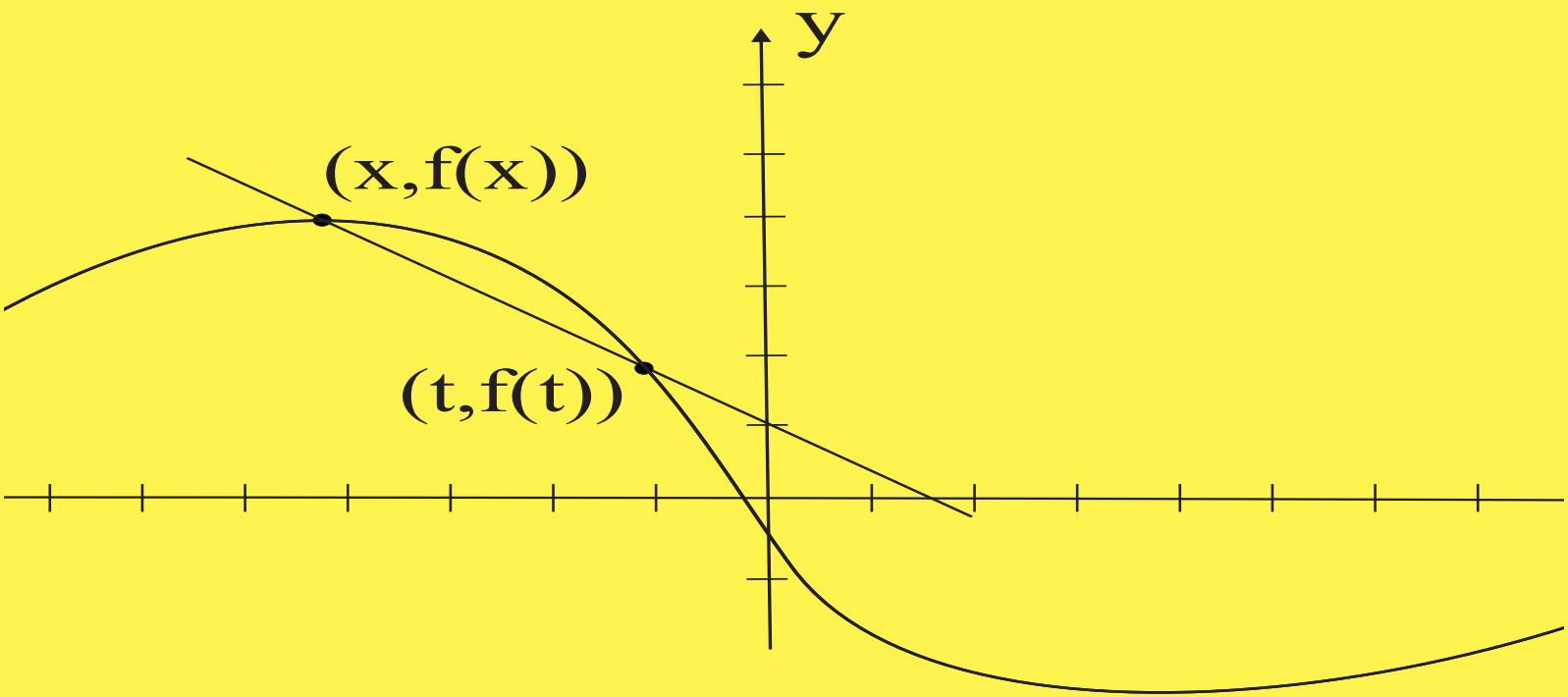
If f is a function with domain an open interval I and if $x \in I$ then the limit

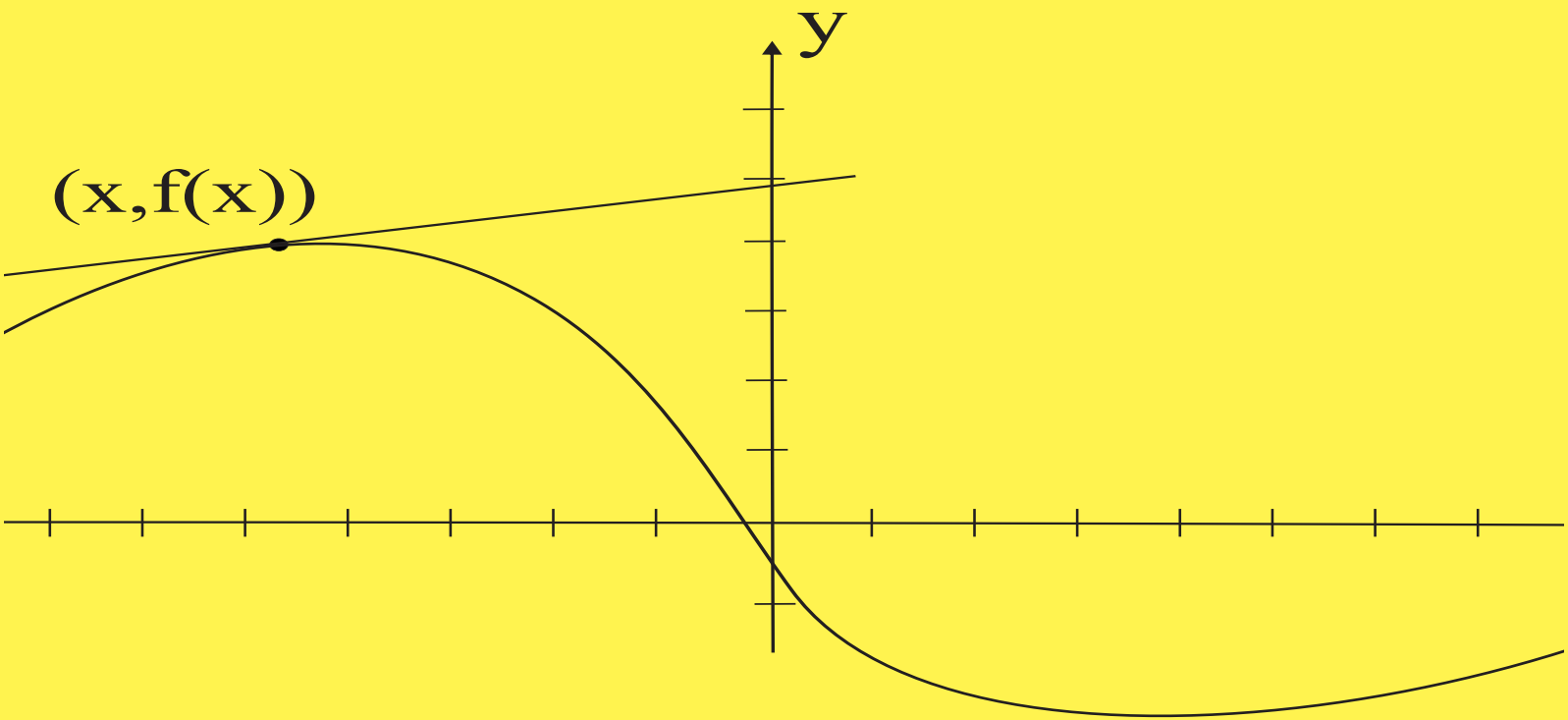
$$\lim_{t \to x} \frac{f(t) - f(x)}{t - x},$$

when it exists, is called the *derivative* of f at x. See the second figure. If the derivative of f at x exists then we say that f is *differentiable* at x. If f is differentiable at every $x \in I$ then we say that f is *differentiable on* I.

We write the derivative of f at x either as

$$f'(x) \quad \text{or} \quad \frac{d}{dx} f \quad \text{or} \quad \frac{df}{dx} \quad \text{or} \quad \dot{f}.$$


y
(x,f(x))
(t,f(t))


y
(x,f(x))

We begin our discussion of the derivative by establishing some basic properties and relating the notion of derivative to continuity.

Lemma

If f is differentiable at a point x then f is continuous at x. In particular, $\lim_{t\to x} f(t) = f(x)$.

Proof: We use a theorem about limits to see that

$$\begin{aligned}
\lim_{t\to x}(f(t)-f(x)) &= \lim_{t\to x}\left((t-x)\cdot\frac{f(t)-f(x)}{t-x}\right)\\
&= \lim_{t\to x}(t-x)\cdot\lim_{t\to x}\frac{f(t)-f(x)}{t-x}\\
&= 0\cdot f'(x)\\
&= 0.
\end{aligned}$$

Therefore $\lim_{t\to x} f(t) = f(x)$ and f is continuous at x. □

Example

All differentiable functions are continuous: differentiability is a stronger property than continuity. Observe that the function $f(x) = |x|$ is continuous at every x but is not differentiable at 0. So continuity does not imply differentiability. Details appear in an example below. □

Theorem

Assume that f and g are functions with domain an open interval I and that f and g are differentiable at $x \in I$. Then $f \pm g$, $f \cdot g$, and f/g are differentiable at x (for f/g we assume that $g(x) \neq 0$). Moreover

(a) $(f \pm g)'(x) = f'(x) \pm g'(x)$;

(b) $(f \cdot g)'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$;

(c) $\left(\dfrac{f}{g}\right)'(x) = \dfrac{g(x) \cdot f'(x) - f(x) \cdot g'(x)}{g^2(x)}$.

Proof: Assertion **(a)** is easy and we leave it as an exercise for you. For **(b)**, we write

$$\begin{aligned}
\lim_{t\to x}\frac{(f\cdot g)(t)-(f\cdot g)(x)}{t-x} &= \lim_{t\to x}\left(\frac{(f(t)-f(x))\cdot g(t)}{t-x}\right.\\
&\qquad\left.+\frac{(g(t)-g(x))\cdot f(x)}{t-x}\right)\\
&= \lim_{t\to x}\left(\frac{(f(t)-f(x))\cdot g(t)}{t-x}\right)\\
&\qquad+\lim_{t\to x}\left(\frac{(g(t)-g(x))\cdot f(x)}{t-x}\right)\\
&= \lim_{t\to x}\left(\frac{(f(t)-f(x))}{t-x}\right)\cdot\left(\lim_{t\to x}g(t)\right)\\
&\qquad+\lim_{t\to x}\left(\frac{(g(t)-g(x))}{t-x}\right)\cdot\left(\lim_{t\to x}f(x)\right)
\end{aligned}$$

where we have used properties of limits. Now the first limit is the derivative of f at x, while the third limit is the derivative of g at x.

Also notice that the limit of $g(t)$ equals $g(x)$ by the lemma. The result is that the last line equals

$$f'(x) \cdot g(x) + g'(x) \cdot f(x),$$

as desired.

To prove **(c)**, write

$$\lim_{t \to x} \frac{(f/g)(t) - (f/g)(x)}{t - x}$$

$$= \lim_{t \to x} \frac{1}{g(t) \cdot g(x)} \left(\frac{f(t) - f(x)}{t - x} \cdot g(x) - \frac{g(t) - g(x)}{t - x} \cdot f(x) \right).$$

The proof is now completed by using our theorem about limits to evaluate the individual limits in this expression. □

Example

That $f(x) = x$ is differentiable follows from

$$\lim_{t \to x} \frac{t - x}{t - x} = 1.$$

Any constant function is differentiable (with derivative identically zero) by a similar argument. It follows from the theorem that any polynomial function is differentiable.

On the other hand, the continuous function $f(x) = |x|$ is *not* differentiable at the point $x = 0$. This is so because

$$\lim_{t \to 0^-} \frac{|t| - |0|}{t - x} = \lim_{t \to 0^-} \frac{-t - 0}{t - 0} = -1, \quad \text{while}$$

$$\lim_{t \to 0^+} \frac{|t| - |0|}{t - x} = \lim_{t \to 0^+} \frac{t - 0}{t - 0} = 1\,.$$

So the required limit does not exist. □

Since the subject of differential calculus is concerned with learning uses of the derivative, it concentrates on functions which *are* differentiable. One comes away from the subject with the impression that most functions are differentiable except at a few isolated points—as is the case with the function $f(x) = |x|$. Indeed this was what the mathematicians of the nineteenth century thought.

Therefore it came as a shock when Karl Weierstrass produced a continuous function that is not differentiable at *any point*.

In a sense that can be made precise, *most* continuous functions are of this nature: their graphs "wiggle" so much that they cannot have a tangent line at any point. Now we turn to an elegant variant of the example of Weierstrass that is due to B. L. van der Waerden (1903–1996).

Theorem

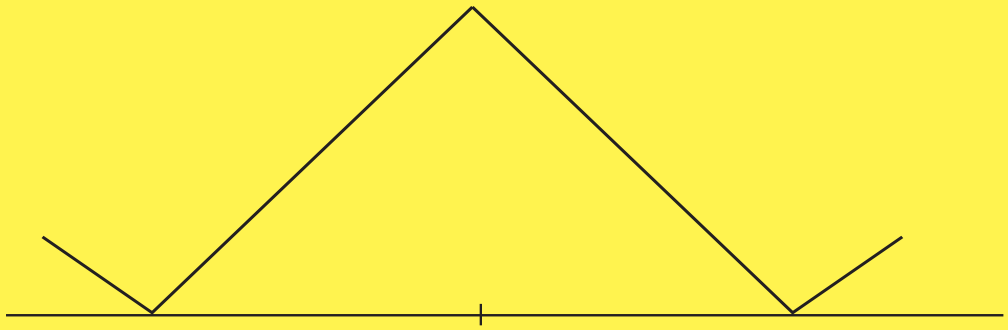
Define a function ψ with domain $\mathbb{R}$ by the rule

$$\psi(x) = \begin{cases} x - n & \text{if} \quad n \leq x < n+1 \text{ and } n \text{ is even} \\ n + 1 - x & \text{if} \quad n \leq x < n+1 \text{ and } n \text{ is odd} \end{cases}$$

for every integer n. The graph of this function is exhibited in the figure on the next slide. Then the function

$$f(x) = \sum_{j=1}^{\infty} \left(\frac{3}{4}\right)^j \psi\left(4^j x\right)$$

is continuous at every real x and differentiable at no real x.


n even
(n + 1) odd

Proof: Since we have not yet discussed series of functions, we take a moment to understand the definition of f. Fix a real x. Notice that $0 \leq \psi(x) \leq 1$ for every x. Then the series becomes a series of numbers, and the jth summand does not exceed $(3/4)^j$ in absolute value. Thus the series converges absolutely; therefore it converges. So it is clear that the displayed formula defines a function of x.

Step I: f is continuous.

To see that f is continuous, pick an $\epsilon > 0$. Choose N so large that

$$\sum_{j=N+1}^{\infty} \left(\frac{3}{4}\right)^j < \frac{\epsilon}{4}$$

(we can of course do this because the series $\sum \left(\frac{3}{4}\right)^j$ converges). Now fix x. Observe that, since ψ is continuous and the graph of ψ is composed of segments of slope 1, we have

$$|\psi(s) - \psi(t)| \leq |s - t|$$

for all s and t. Moreover $|\psi(s) - \psi(t)| \leq 1$ for all s, t.

For $j = 1, 2, \ldots, N$, pick $\delta_j > 0$ so that, when $|t - x| < \delta_j$, then

$$\left|\psi\left(4^j t\right) - \psi\left(4^j x\right)\right| < \frac{\epsilon}{8}.$$

Let δ be the minimum of $\delta_1, \ldots \delta_N$.

Now, if $|t-x|<\delta$, then

$$\begin{aligned}
|f(t)-f(x)| &= \left|\sum_{j=1}^{N}\left(\frac{3}{4}\right)^j\cdot\left(\psi(4^jt)-\psi(4^jx)\right)\right.\\
&\qquad\left.+\sum_{j=N+1}^{\infty}\left(\frac{3}{4}\right)^j\cdot\left(\psi(4^jt)-\psi(4^jx)\right)\right|\\
&\le \sum_{j=1}^{N}\left(\frac{3}{4}\right)^j\left|\left(\psi(4^jt)-\psi(4^jx)\right)\right|\\
&\qquad+\sum_{j=N+1}^{\infty}\left(\frac{3}{4}\right)^j\left|\psi(4^jt)-\psi(4^jx)\right|\\
&\le \sum_{j=1}^{N}\left(\frac{3}{4}\right)^j\cdot\frac{\epsilon}{8}+\sum_{j=N+1}^{\infty}\left(\frac{3}{4}\right)^j\cdot
\end{aligned}$$

Here we have used the choice of δ to estimate the summands in the first sum. The first sum is thus less than $\epsilon/2$ (just notice that $\sum_{j=1}^{\infty}(3/4)^j < 4$). The second sum is less than $\epsilon/2$ by the choice of N. Altogether then

$$|f(t) - f(x)| < \epsilon$$

whenever $|t - x| < \delta$. Therefore f is continuous, indeed uniformly so.

Step II: f is nowhere differentiable.

Fix x. For $\ell = 1, 2, \ldots$ define $t_\ell = x \pm 4^{-\ell}/2$. We will say whether the sign is plus or minus in a moment (this will depend on the position of x relative to the integers). Then

$$\left|\frac{f(t_\ell) - f(x)}{t_\ell - x}\right| = \left|\frac{1}{t_\ell - x}\left[\sum_{j=1}^{\ell}\left(\frac{3}{4}\right)^j \left(\psi(4^j t_\ell) - \psi(4^j x)\right)\right.\right.$$
$$\left.\left. + \sum_{j=\ell+1}^{\infty}\left(\frac{3}{4}\right)^j \left(\psi(4^j t_\ell) - \psi(4^j x)\right)\right]\right|. \qquad (1)$$

Notice that, when $j \geq \ell + 1$, then $4^j t_\ell$ and $4^j x$ differ by an even integer. Since ψ has period 2, we find that each of the summands in the second sum is 0. Next we turn to the first sum.

We choose the sign—plus or minus—in the definition of t_ℓ so that there is no integer lying between $4^\ell t_\ell$ and $4^\ell x$. We can do this because the two numbers differ by $1/2$. But then the ℓth summand has magnitude

$$(3/4)^\ell \cdot |4^\ell t_\ell - 4^\ell x| = 3^\ell |t_\ell - x| .$$

On the other hand, the first $\ell - 1$ summands add up to not more than

$$\sum_{j=1}^{\ell-1} \left(\frac{3}{4}\right)^j \cdot |4^j t_\ell - 4^j x| = \sum_{j=1}^{\ell-1} 3^j \cdot 4^{-\ell}/2 \le \frac{3^\ell - 1}{3-1} \cdot 4^{-\ell}/2 \le 3^\ell \cdot 4^{-\ell-1} .$$

It follows that

$$\begin{aligned}
\left|\frac{f(t_\ell)-f(x)}{t_\ell-x}\right| &= \frac{1}{|t_\ell-x|}\cdot\left|\sum_{j=1}^{\ell}\left(\frac{3}{4}\right)^j\left(\psi(4^j t_\ell)-\psi(4^j x)\right)\right| \\
&= \frac{1}{|t_\ell-x|}\left|\cdot\sum_{j=1}^{\ell-1}\left(\frac{3}{4}\right)^j\left(\psi(4^j t_\ell)-\psi(4^j x)\right)\right. \\
&\qquad \left.+\left(\frac{3}{4}\right)^\ell\left(\psi(4^\ell t_\ell)-\psi(4^\ell x)\right)\right|
\end{aligned}$$

$$\begin{aligned}
&\geq \frac{1}{|t_\ell - x|} \cdot \left| \left(\frac{3}{4}\right)^\ell \psi(4^\ell t_\ell) - \left(\frac{3}{4}\right)^\ell \psi(4^\ell x) \right| \\
&\quad - \frac{1}{|t_\ell - x|} \left| \sum_{j=1}^{\ell-1} \left(\frac{3}{4}\right)^j \left(\psi(4^j t_\ell) - \psi(4^j x)\right) \right| \\
&\geq 3^\ell - \frac{1}{(4^{-\ell}/2)} \cdot 3^\ell \cdot 4^{-\ell-1} \geq 3^{\ell-1}.
\end{aligned}$$

Thus $t_\ell \to x$ but the Newton quotients blow up as $\ell \to \infty$. Therefore the limit

$$\lim_{t \to x} \frac{f(t) - f(x)}{t - x}$$

cannot exist. The function f is not differentiable at x. □

The proof of the last theorem was long, but the idea is simple: the function f is built by piling oscillations on top of oscillations. When the ℓth oscillation is added, it is made very small in size so that it does not cancel the previous oscillations. But it is made very steep so that it will cause the derivative to become large.

The practical meaning of Weierstrass's example is that we should realize that differentiability is a very strong and special property of functions. Most continuous functions are not differentiable at any point. When we are proving theorems about continuous functions, we should *not* think of them in terms of properties of differentiable functions.

Next we turn to the Chain Rule.

Theorem

Let g be a differentiable function on an open interval I and let f be a differentiable function on an open interval that contains the range of g. Then $f \circ g$ is differentiable on the interval I and

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$$

for each $x \in I$.

Proof: We use the notation Δt to stand for an increment in the variable t. Let us use the symbol $\mathcal{V}(r)$ to stand for any expression which tends to 0 as $\Delta r \to 0$. Fix $x \in I$. Set $r = g(x)$. By hypothesis,

$$\lim_{\Delta r \to 0} \frac{f(r + \Delta r) - f(r)}{\Delta r} = f'(r)$$

or

$$\frac{f(r+\Delta r)-f(r)}{\Delta r}-f'(r)=\mathcal{V}(r)$$

or

$$f(r + \Delta r) = f(r) + \Delta r \cdot f'(r) + \Delta r \cdot \mathcal{V}(r)\,.$$

Notice that this equation is valid even when $\Delta r = 0$. Since Δr in the equation can be any small quantity, we set

$$\Delta r = \Delta x \cdot [g'(x) + \mathcal{V}(x)]\,.$$

Substituting this expression into the equation and using the fact that $r = g(x)$ yields

$$\begin{aligned} & f(g(x) + \Delta x[g'(x) + \mathcal{V}(x)]) \\ =& f(r) + (\Delta x \cdot [g'(x) + \mathcal{V}(x)]) \cdot f'(r) + (\Delta x \cdot [g'(x) + \mathcal{V}(x)]) \cdot \mathcal{V}(r) \\ =& f(g(x)) + \Delta x \cdot f'(g(x)) \cdot g'(x) + \Delta x \cdot \mathcal{V}(x). \end{aligned} \tag{2}$$

Just as we derived above, we may also obtain

$$\begin{aligned} g(x+\Delta x) &= g(x) + \Delta x \cdot g'(x) + \Delta x \cdot \mathcal{V}(x) \\ &= g(x) + \Delta x[g'(x) + \mathcal{V}(x)]\,. \end{aligned}$$

We may substitute this equality into the left side of our last calculation to obtain

$$f(g(x+\Delta x)) = f(g(x)) + \Delta x \cdot f'(g(x)) \cdot g'(x) + \Delta x \cdot \mathcal{V}(x)\,.$$

With some algebra this can be rewritten as

$$\frac{f(g(x+\Delta x)) - f(g(x))}{\Delta x} - f'(g(x)) \cdot g'(x) = \mathcal{V}(x)\,.$$

But this just says that

$$\lim_{\Delta x \to 0} \frac{(f \circ g)(x+\Delta x) - (f \circ g)(x)}{\Delta x} = f'(g(x)) \cdot g'(x)\,.$$

That is, $(f \circ g)'(x)$ exists and equals $f'(g(x)) \cdot g'(x)$, as desired. □

Example

The derivative of

$$f(x) = \sin(x^3 - x^2)$$

is

$$f'(x) = [\cos(x^3 - x^2)] \cdot (3x^2 - 2x)\,.$$ □