Math 4111
November 16, 2020 Lecture

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Figure: This is your instructor.

We begin by refining our notion of limit:

Definition

Fix $P \in \mathbb{R}$. Let f be a function with domain E. Suppose that P is a limit point of $E \cap [P-1, P)$. We say that f has *left limit* ℓ at P, and write

$$\lim_{x \to P^-} f(x) = \ell$$

if, for every $\epsilon > 0$, there is a $\delta > 0$ such that, whenever $x \in E$ and $P - \delta < x < P$, then it holds that

$$|f(x) - \ell| < \epsilon \, .$$

Now suppose that P is a limit point of $E \cap (P, P+1]$. We say that f has *right limit* m at P, and write

$$\lim_{x \to P^+} f(x) = m$$

if, for every $\epsilon > 0$, there is a $\delta > 0$ such that, whenever $x \in E$ and $P < x < P + \delta$, then it holds that

$$|f(x) - m| < \epsilon\,.$$

This definition simply formalizes the notion of either letting x tend to P from the left only or from the right only.

Example

Let

$$f(x) = \begin{cases} 0 & \text{if} \quad x \leq 0 \\ \sin(1/x) & \text{if} \quad x > 0\,. \end{cases}$$

Then $\lim_{x \to 0-} f(x) = 0$ and $\lim_{x \to 0+} f(x)$ does not exist. □

Definition

Fix $P \in \mathbb{R}$. Let f be a function with domain E. Suppose that P is a limit point of $E \cap [P-1, P)$ and that P is an element of E. We say that f is *left continuous* at P if

$$\lim_{x \to P^-} f(x) = f(P).$$

Likewise, in case P is a limit point of $E \cap (P, P+1]$ and is also an element of E, we say that f is *right continuous* at P if

$$\lim_{x \to P^+} f(x) = f(P).$$

Example

Define

$$f(x) = \begin{cases} 1 & \text{if} \quad x < 0 \\ 0 & \text{if} \quad x = 0 \\ x \sin(1/x) & \text{if} \quad x > 0\,. \end{cases}$$

Then f is right continuous at 0 but f is not left continuous at 0.

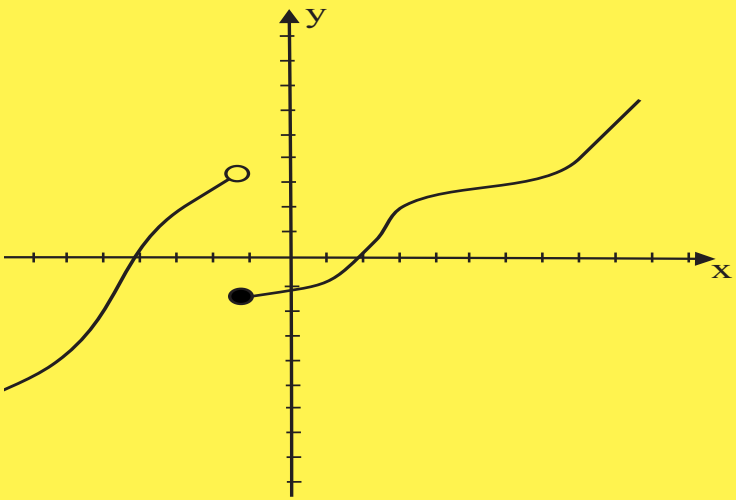
□

Let f be a function with domain E. Let P in E and assume that f is discontinuous at P. There are two ways in which this discontinuity can occur:

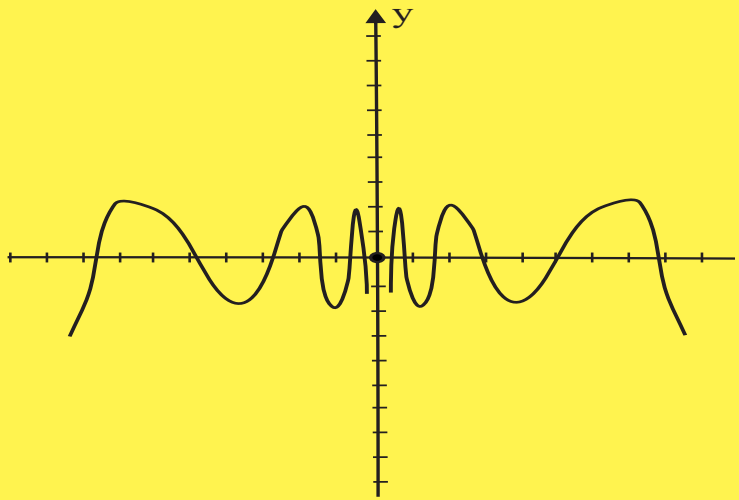
I. If $\lim_{x \to P^-} f(x)$ and $\lim_{x \to P^+} f(x)$ both exist but either do not equal each other or do not equal $f(P)$ then we say that f has a *discontinuity of the first kind* (or a *simple discontinuity*, or *a jump discontinuity*) at P.

II. If either $\lim_{x \to P^-} f(x)$ does not exist or $\lim_{x \to P^+} f(x)$ does not exist then we say that f has a *discontinuity of the second kind* at P.

Refer to Figure on the next slide.



continuity of the first kind



discontinuity of the second

Example

Define

$$f(x) = \begin{cases} \sin(1/x) & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad x = 0 \end{cases}$$

$$g(x) = \begin{cases} 1 & \text{if} \quad x > 0 \\ 0 & \text{if} \quad x = 0 \\ -1 & \text{if} \quad x < 0 \end{cases}$$

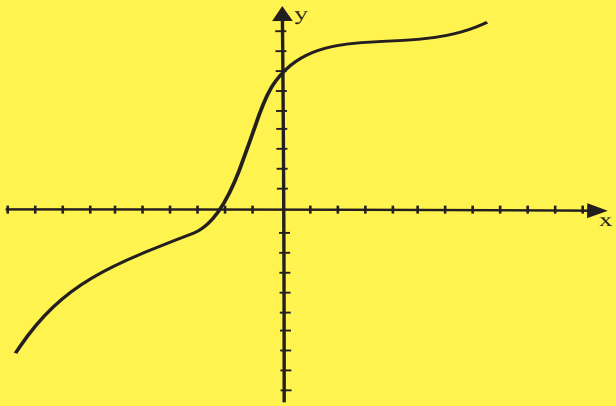
$$h(x) = \begin{cases} 1 & \text{if } x \text{ is irrational} \\ 0 & \text{if } x \text{ is rational} \end{cases}$$

Then f has a discontinuity of the second kind at 0 while g has a discontinuity of the first kind at 0. The function h has a discontinuity of the second kind at every point. □

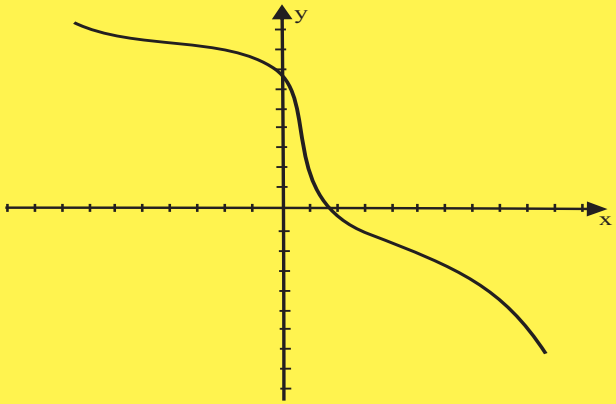
Definition

Let f be a function whose domain contains an open interval (a, b). We say that f is *increasing* on (a, b) if, whenever $a < s < t < b$, it holds that $f(s) \leq f(t)$. We say that f is *decreasing* on (a, b) if, whenever $a < s < t < b$, it holds that $f(s) \geq f(t)$. See the figure below. If a function is either increasing or decreasing then we call it *monotone* or *monotonic*. Compare with the definition of monotonic sequences in Section 2.1.

As with sequences, the word "monotonic" is superfluous in many contexts. But its use is traditional and occasionally convenient.



monotone increasing function



monotone decreasing function

Proposition

Let f be a monotonic function on an open interval (a, b). Then all of the discontinuities of f are of the first kind.

Proof: It is enough to show that, for each $P \in (a, b)$, the limits

$$\lim_{x \to P^-} f(x)$$

and

$$\lim_{x \to P^+} f(x)$$

exist.

Let us first assume that f is monotonically increasing. Fix $P \in (a, b)$. If $a < s < P$ then $f(s) \leq f(P)$. Therefore $S = \{f(s) : a < s < P\}$ is bounded above. Let M be the least upper bound of S.

Pick $\epsilon > 0$. By definition of least upper bound there must be an $f(s) \in S$ such that $|f(s) - M| < \epsilon$. Let $\delta = |P - s|$. If $P - \delta < t < P$ then $s < t < P$ and $f(s) \leq f(t) \leq M$ or $|f(t) - M| < \epsilon$. Thus $\lim_{x \to P^-} f(x)$ exists and equals M.

If we set m equal to the infimum of the set $T = \{f(t) : P < t < b\}$ then a similar argument shows that $\lim_{x \to P^+} f(x)$ exists and equals m. That completes the proof.

The argument in case f is monotonically decreasing is just the same, and we omit the details. □

Corollary

Let f be a monotonic function on an interval (a, b). Then f has at most countably many discontinuities.

Proof: Assume for simplicity that f is monotonically increasing. If P is a discontinuity then the proposition tells us that

$$\lim_{x \to P^-} f(x) < \lim_{x \to P^+} f(x).$$

Therefore there is a rational number q_P between $\lim_{x \to P^-} f(x)$ and $\lim_{x \to P^+} f(x)$. Notice that different discontinuities will have different rational numbers associated to them because if $\widehat{P}$ is another discontinuity and, say, $\widehat{P} < P$ then

$$\lim_{x \to \widehat{P}^-} f(x) < q_{\widehat{P}} < \lim_{x \to \widehat{P}^+} f(x) \leq \lim_{x \to P^-} f(x) < q_P < \lim_{x \to P^+} f(x).$$

Thus we have exhibited a one-to-one function from the set of discontinuities of f into the set of rational numbers. It follows that the set of discontinuities is countable.

The argument in case f is monotonically decreasing is just the same, and we omit the details. □

A continuous function f has the property that the inverse image under f of any open set is open. However, it is not in general true that the *image* under f of any open set is open. A counterexample is the function $f(x) = x^2$ and the open set $\mathcal{O} = (-1, 1)$ whose image under f is $[0, 1)$.

Example

Consider the greatest integer function $f(x) = [x]$. This means that $f(x)$ equals the greatest integer which is less than or equal to x. Then f is monotone increasing, and its discontinuities are of the first kind and are at the integers. This example illustrates the preceding Proposition and Corollary. □

Definition

Suppose that f is a function on (a, b) such that $a < s < t < b$ implies $f(s) < f(t)$. Such a function is called *strictly increasing* (*strictly decreasing* functions are defined similarly). We refer to such functions as *strictly monotone*.

It is clear that a strictly increasing (resp. decreasing) function is one-to-one, hence has an inverse. Now we prove:

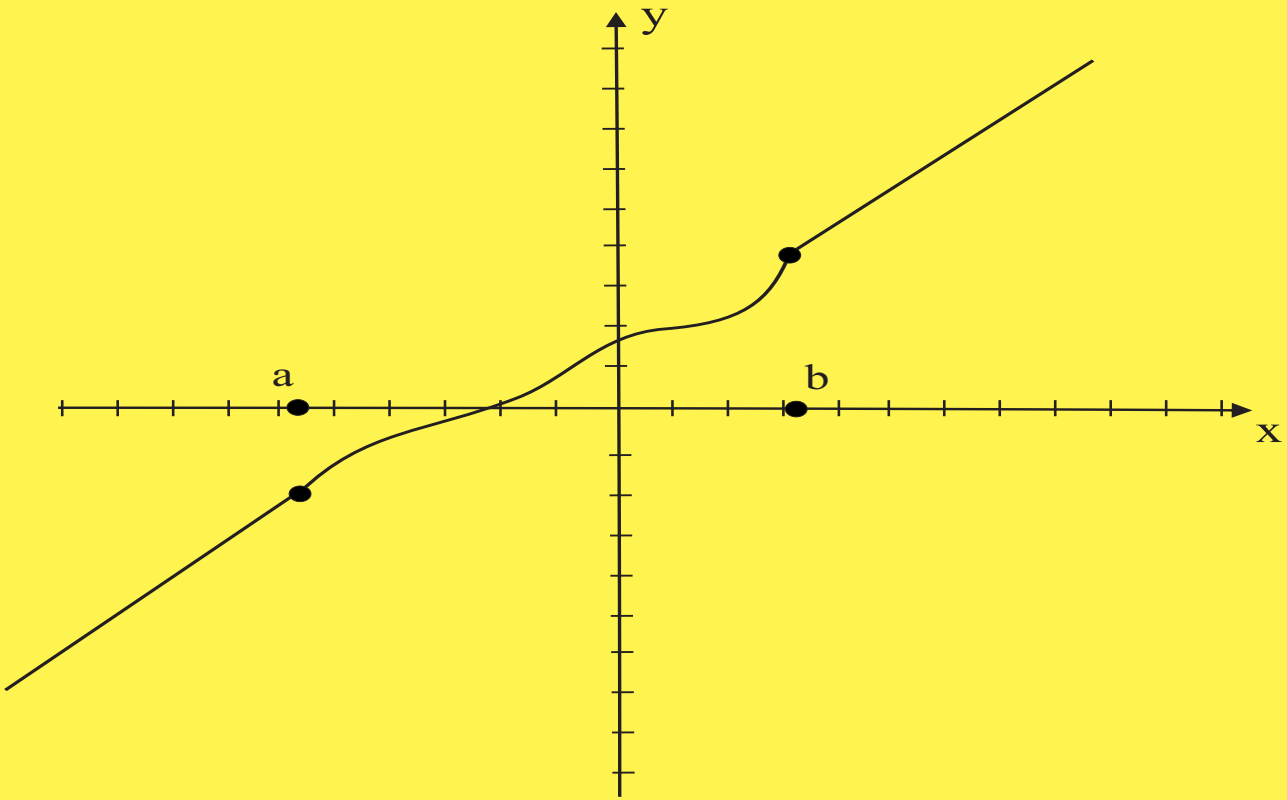
Theorem

Let f be a strictly monotone, continuous function with domain $[a, b]$. Then f^{-1} exists and is continuous.

Proof: Assume without loss of generality that f is strictly monotone *increasing*. Let us extend f to the entire real line by defining

$$f(x) = \begin{cases} (x-a) + f(a) & \text{if} \quad x < a \\ \text{as given} & \text{if} \quad a \leq x \leq b \\ (x-b) + f(b) & \text{if} \quad x > b\,. \end{cases}$$

See the figure below. Then it is easy to see that this extended version of f is still continuous and is strictly monotone increasing on all of $\mathbb{R}$.


y
x
a
b

That f^{-1} exists has already been discussed. The extended function f takes any open interval (c, d) to the open interval $(f(c), f(d))$. Since any open set is a union of open intervals, we see that f takes any open set to an open set. In other words, $\left[f^{-1}\right]^{-1}$ takes open sets to open sets. But this just says that f^{-1} is continuous.

Since the inverse of the extended function f is continuous, then so is the inverse of the original function f. That completes the proof. $\square$

Example

Consider the function

$$f(x) = e^x \, .$$

It is strictly inreasing on the entire real line, so it has an inverse. Its inverse is in fact the natural logarithm function $\ln x$.

Now take a look at the function

$$g(x) = \begin{cases} x & \text{if} \quad x \leq -1 \\ -1 & \text{if} \quad -1 < x < 1 \\ x - 2 & \text{if} \quad 1 \leq x \, . \end{cases}$$

This is a monotone increasing function, but it is *not* strictly increasing. And it has no inverse because it is not one-to-one. □