Math 4111
November 9, 2020 Lecture

Steven G. Krantz



Figure: This is your instructor.

Recall that in Chapter 4 we learned a characterization of compact sets in terms of open covers. In Section 2 of the present chapter we learned a characterization of continuous functions in terms of inverse images of open sets. Thus it is not surprising that compact sets and continuous functions interact in a natural way. We explore this interaction in the present section.

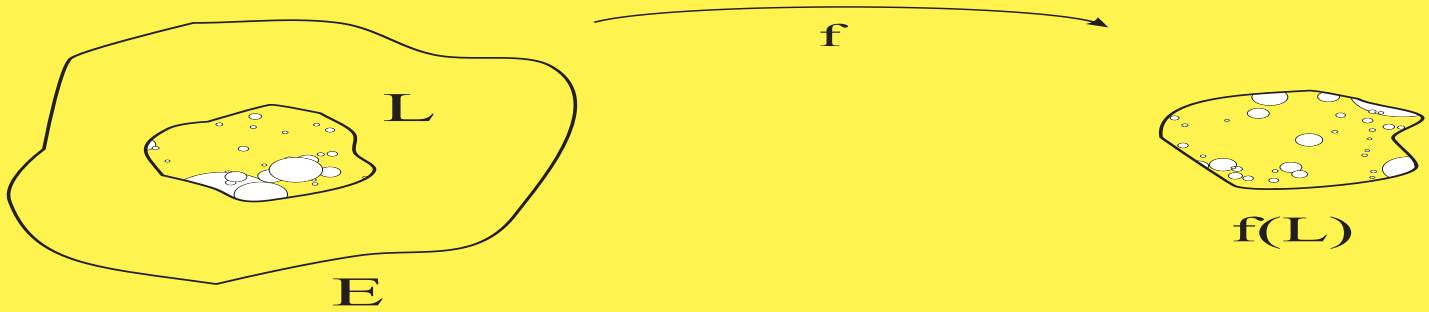


Figure: The image of the set L under the function f.

Definition

Let f be a function with domain E and let L be a subset of E. We define

$$f(L) = \{f(x) : x \in L\}.$$

The set $f(L)$ is called the *image* of L under f. See the figure above.

Theorem

The image of a compact set under a continuous function is also compact.

Proof: Let f be a continuous function with domain E and let K be a subset of E that is compact. Our job is to show that $f(K)$ is compact.

Let $\mathcal{C} = \{\mathcal{O}_\alpha\}$ be an open covering of $f(K)$. Since f is continuous we know that, for each α, the set $f^{-1}(\mathcal{O}_\alpha)$ is the intersection of E with an open set $\mathcal{U}_\alpha$. Let $\widehat{\mathcal{C}} = \{\mathcal{U}_\alpha\}_{\alpha \in A}$. Since $\mathcal{C}$ covers $f(K)$ it follows that $\widehat{\mathcal{C}}$ covers K. But K is compact; therefore (Theorem 4.37) there is a finite subcovering

$$\{\mathcal{U}_{\alpha_1}, \mathcal{U}_{\alpha_2}, \ldots \mathcal{U}_{\alpha_m}\}$$

of K. But then it follows that $f(\mathcal{U}_{\alpha_1} \cap E), \ldots, f(\mathcal{U}_{\alpha_m} \cap E)$ covers $f(K)$, hence

$$\mathcal{O}_{\alpha_1}, \mathcal{O}_{\alpha_2}, \ldots, \mathcal{O}_{\alpha_m}$$

covers $f(K)$.
We have taken an arbitrary open cover $\mathcal{C}$ for $f(K)$ and extracted from it a finite subcovering. It follows that $f(K)$ is compact. $\square$

It is not the case that the continuous image of a closed set is closed. For instance, take $f(x) = 1/(1+x^2)$ and $E = \mathbb{R}$: the set E is closed and f is continuous but $f(E) = (0, 1]$ is not closed.

It is also not the case that the continuous image of a bounded set is bounded. As an example, take $f(x) = 1/x$ and $E = (0, 1)$. Then E is bounded and f continuous but $f(E) = (1, \infty)$ is unbounded.

However, the combined properties of closedness *and* boundedness (that is, compactness) are preserved. That is the content of the preceding theorem.

Corollary

Let f be a continuous, real-valued function with compact domain $K \subseteq \mathbb{R}$. Then there is a number L such that

$$|f(x)| \leq L$$

for all $x \in K$.

Proof: We know from the theorem that $f(K)$ is compact. By Theorem 4.29, we conclude that $f(K)$ is bounded. Thus there is a number L such that $|t| \leq L$ for all $t \in f(K)$. But that is just the assertion that we wish to prove. □

In fact we can prove an important strengthening of the corollary. Since $f(K)$ is compact, it contains its supremum M and its infimum m. Therefore there must be a number $C \in K$ such that $f(C) = M$ and a number $c \in K$ such that $f(c) = m$. In other words, $f(c) \leq f(x) \leq f(C)$ for all $x \in K$. We summarize:

Theorem

Let f be a continuous function on a compact set $K \subseteq \mathbb{R}$. Then there exist numbers c and C in K such that $f(c) \leq f(x) \leq f(C)$ for all $x \in K$. We call c an absolute minimum for f on K and C an absolute maximum for f on K. We call $f(c)$ the absolute minimum value for f on K and $f(C)$ the absolute maximum value for f on K.

Notice that, in the last theorem, the location of the absolute maximum and absolute minimum need not be unique. For instance, the function $\sin x$ on the compact interval $[0, 4\pi]$ has an absolute minimum at $3\pi/2$ and $7\pi/2$. It has an absolute maximum at $\pi/2$ and at $5\pi/2$.

Now we define a refined type of continuity called "uniform continuity." We shall learn that this new notion of continuous function arises naturally for a continuous function on a compact set. It will also play an important role in our later studies, especially in the context of the integral.

Definition

Let f be a function with domain $E \subseteq \mathbb{R}$. We say that f is *uniformly continuous* on E if, for each $\epsilon > 0$, there is a $\delta > 0$ such that, whenever $s, t \in E$ and $|s - t| < \delta$, then $|f(s) - f(t)| < \epsilon$.

Observe that "uniform continuity" differs from "continuity" in that it treats all points of the domain simultaneously: the $\delta > 0$ that is chosen is independent of the points $s, t \in E$. This difference is highlighted by the next two examples.

Example

Suppose that a function $f : \mathbb{R} \to \mathbb{R}$ satisfies the condition

$$|f(s) - f(t)| \leq C \cdot |s - t|\,, \tag{5.25.1}$$

where C is some positive constant. This is called a *Lipschitz condition*, and it arises frequently in analysis. Let $\epsilon > 0$ and set $\delta = \epsilon / C$. If $|x - y| < \delta$ then, by (5.25.1),

$$|f(x) - f(y)| \leq C \cdot |x - y| < C \cdot \delta = C \cdot \frac{\epsilon}{C} = \epsilon\,.$$

It follows that f is uniformly continuous. □

Example

Consider the function $f(x) = x^2$. Fix a point $P \in \mathbb{R}, P > 0$, and let $\epsilon > 0$. In order to guarantee that $|f(x) - f(P)| < \epsilon$ we must have (for $x > 0$)

$$|x^2 - P^2| < \epsilon$$

or

$$|x - P| < \frac{\epsilon}{x + P}.$$

Since x will range over a neighborhood of P, we see that the required δ in the definition of continuity cannot be larger than $\epsilon/(2P)$. In fact the choice $|x - P| < \delta = \epsilon/(2P + 1)$ will do the job.

Put in slightly different words, let $\epsilon = 1$. Then $|f(j + 1/j) - f(j)| > \epsilon = 1$ for any j. Thus, for this ϵ, we may not take δ to be $1/j$ for any j. So no uniform δ exists.
Thus the choice of δ depends not only on ϵ (which we have come to expect) but also on P. In particular, f is not uniformly continuous on $\mathbb{R}$. This is a quantitative reflection of the fact that the graph of f becomes ever steeper as the variable x moves to the right.

Notice that the same calculation shows that the function f with restricted domain $[a, b], 0 < a < b < \infty$, *is* uniformly continuous. That is because, when the function is restricted to $[a, b]$, its slope does not become arbitrarily large. See the figure below. □

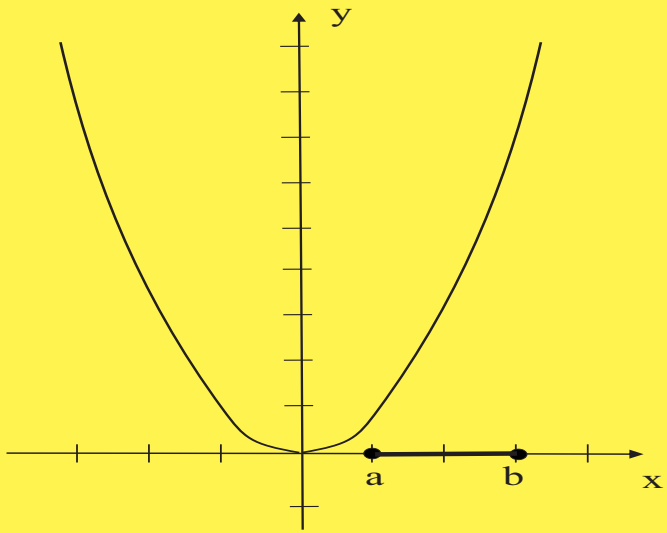


Figure: Uniform continuity on the interval $[a, b]$.

Now the main result about uniform continuity is the following:

Theorem

Let f be a continuous function with compact domain K. Then f is uniformly continuous on K.

Proof: Pick $\epsilon > 0$. By the definition of continuity there is for each point $x \in K$ a number $\delta_x > 0$ such that if $|x - t| < \delta_x$ then $|f(t) - f(x)| < \epsilon/2$. The intervals $I_x = (x - \delta_x/2, x + \delta_x/2)$ form an open covering of K. Since K is compact, we may therefore (by Theorem 4.34) extract a finite subcovering

$$I_{x_1}, \ldots I_{x_m}.$$

Now let $\delta = \min\{\delta_{x_1}/2, \ldots, \delta_{x_m}/2\} > 0$. If $s, t \in K$ and $|s - t| < \delta$ then $s \in I_{x_j}$ for some $1 \leq j \leq m$. It follows that

$$|s - x_j| < \delta_{x_j}/2$$

and

$$|t - x_j| \leq |t - s| + |s - x_j| < \delta + \delta_{x_j}/2 \leq \delta_{x_j}/2 + \delta_{x_j}/2 = \delta_{x_j}\,.$$

We know that

$$|f(s) - f(t)| \leq |f(s) - f(x_j)| + |f(x_j) - f(t)|\,.$$

But since each of s and t is within δ_{x_j} of x_j we may conclude that the last line is less than

$$\frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon\,.$$

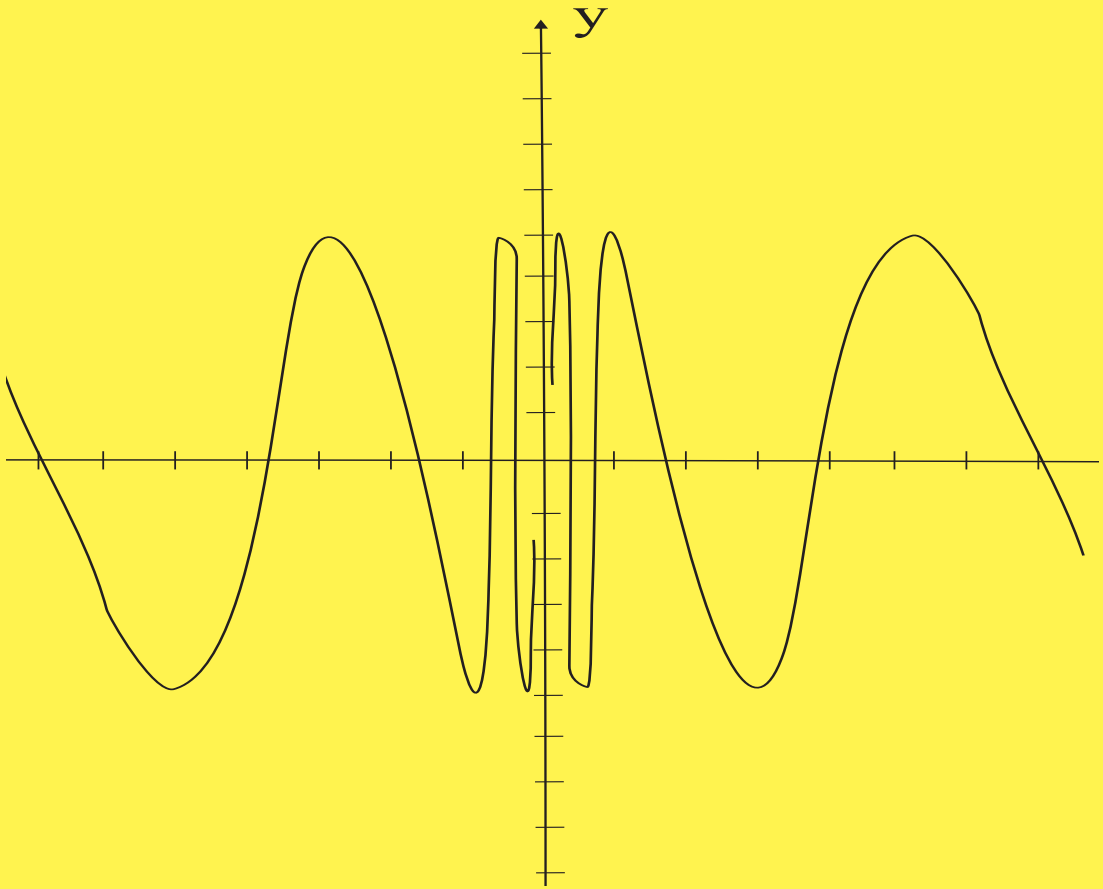
Notice that our choice of δ does not depend on s and t (indeed, we chose δ *before* we chose s and t). We conclude that f is uniformly continuous.

□

Remark. Where in the proof did the compactness play a role? We defined δ to be the minimum of $\delta_{x_1}, \ldots \delta_{x_m}$. In order to guarantee that δ be *positive* it is crucial that we be taking the minimum of *finitely many* positive numbers. So we needed a *finite* subcovering.

Example

The function $f(x) = \sin(1/x)$ is continuous on the domain $E = (0, \infty)$ since it is the composition of continuous functions (see the figure below).



However, it is not uniformly continuous since

$$\left| f\left(\frac{1}{2j\pi}\right) - f\left(\frac{1}{\frac{(4j+1)\pi}{2}}\right) \right| = 1$$

for $j = 1, 2, \ldots$. Thus, even though the arguments are becoming arbitrarily close together, the images of these arguments remain bounded apart. We conclude that f cannot be uniformly continuous.

However, if f is considered as a function on any interval of the form $[a, b], 0 < a < b < \infty$, then the preceding theorem tells us that the function f is uniformly continuous. □

As an exercise, you should check that

$$g(x) = \begin{cases} x \sin(1/x) & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad x = 0 \end{cases}$$

is uniformly continuous on any interval of the form $[-N, N]$. See the figure below.

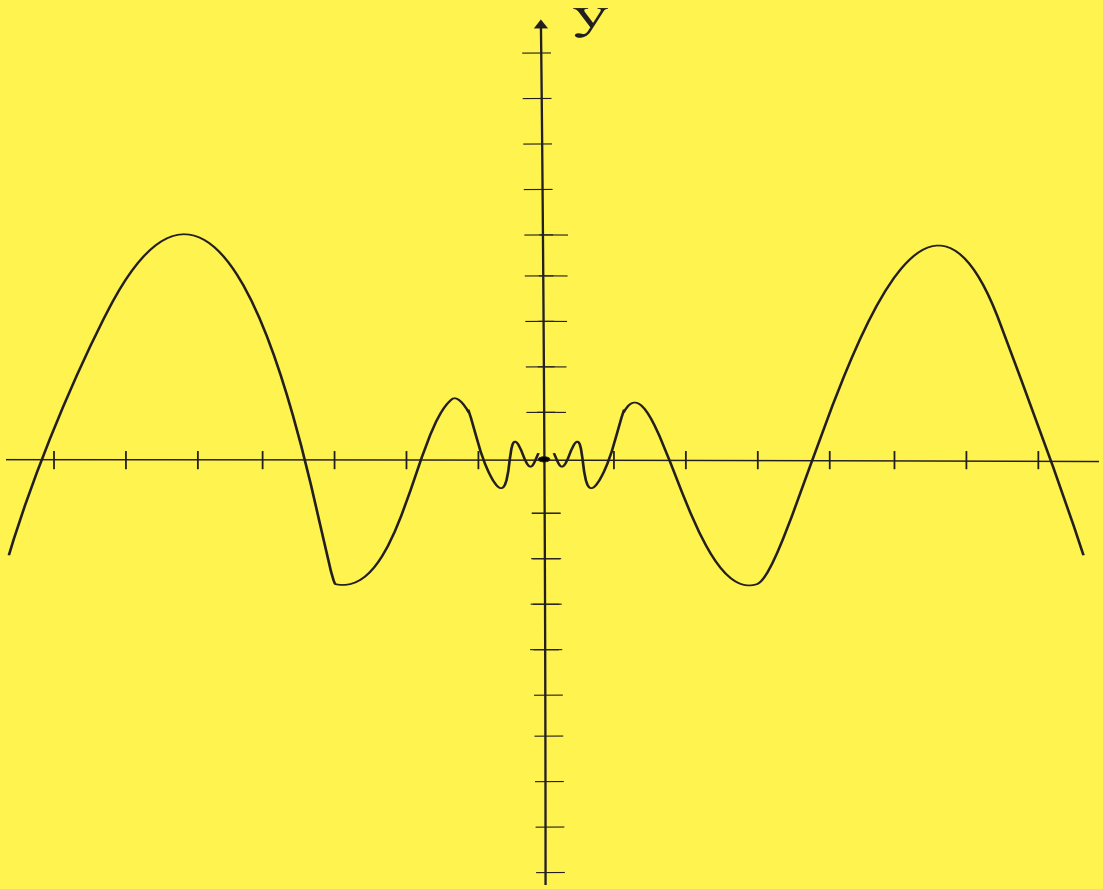


Figure: Uniform continuity on the interval $[a, b]$.

Next we show that continuous functions preserve connectedness.

Theorem

Let f be a continuous function with domain an open interval I. Suppose that L is a connected subset of I. Then $f(L)$ is connected.

Proof: Suppose to the contrary that there are open sets U and V such that

$$U \cap f(L) \neq \emptyset, V \cap f(L) \neq \emptyset,$$

$$(U \cap f(L)) \cap (V \cap f(L)) = \emptyset,$$

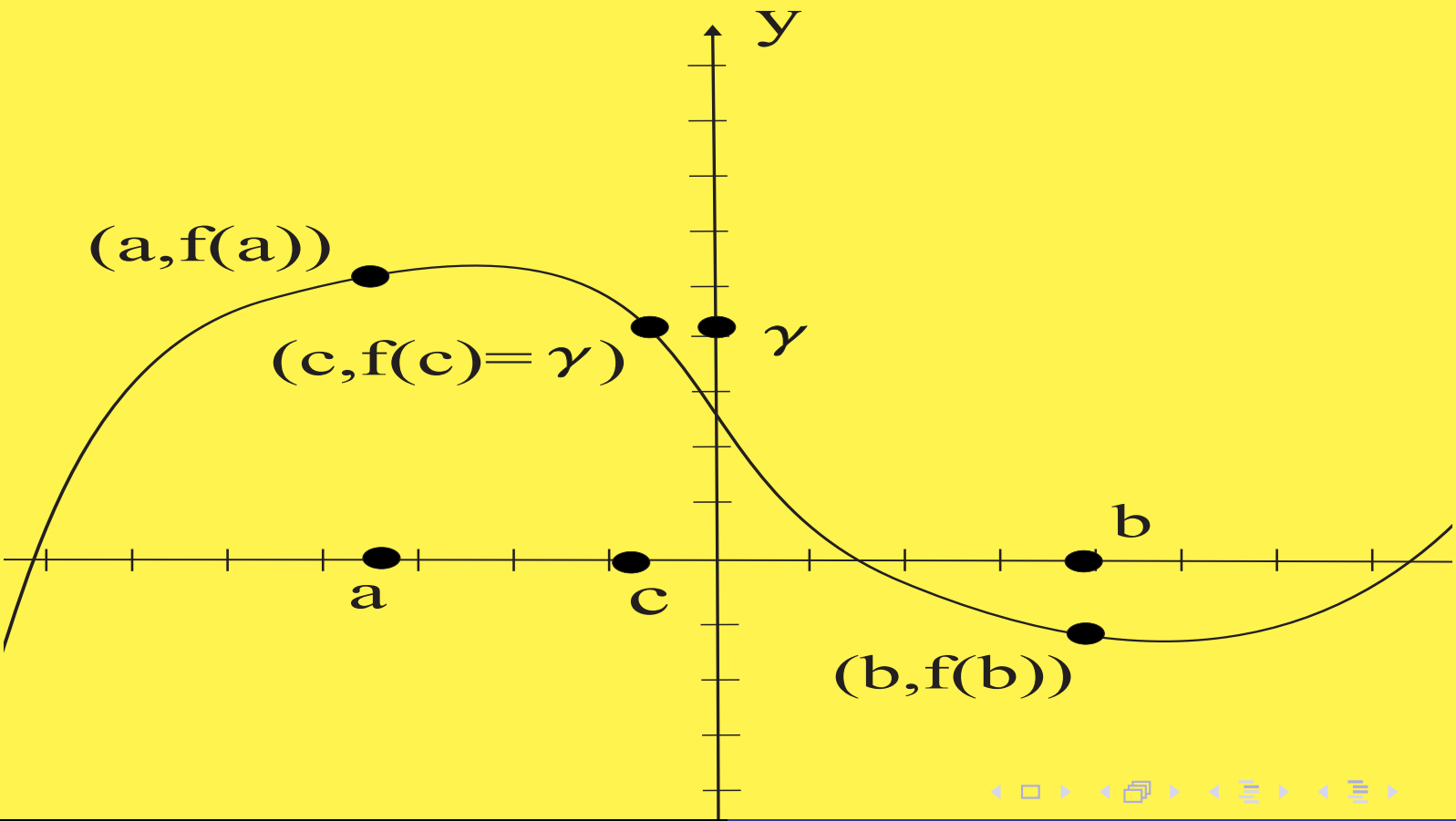
and

$$f(L) = (U \cap f(L)) \cup (V \cap f(L)) .$$

Since f is continuous, $f^{-1}(U)$ and $f^{-1}(V)$ are open. They each have nonempty intersection with L since $U \cap f(L)$ and $V \cap f(L)$ are nonempty. By the definition of f^{-1}, they are disjoint. And since $U \cup V$ contains $f(L)$ it follows, by definition, that $f^{-1}(U) \cup f^{-1}(V)$ contains L. But this shows that L is disconnected, and that is a contradiction. □

Corollary (The Intermediate Value Theorem)

Let f be a continuous function whose domain contains the interval $[a, b]$. Let γ be a number that lies between $f(a)$ and $f(b)$. Then there is a number c between a and b such that $f(c) = \gamma$. Refer to Figure 5.7.



Proof: The set $[a, b]$ is connected. Therefore $f([a, b])$ is connected. But $f([a, b])$ contains the points $f(a)$ and $f(b)$. By connectivity, $f([a, b])$ must contain the interval that has $f(a)$ and $f(b)$ as endpoints. In particular, $f([a, b])$ must contain any number γ that lies between $f(a)$ and $f(b)$. But this just says that there is a number c lying between a and b such that $f(c) = \gamma$. That is the desired conclusion. $\square$

Example

Let f be a continuous function with domain the interval $[0, 1]$ and range the interval $[0, 1]$. We claim that f has a fixed point, that is, a point p such that $f(p) = p$.
To see this, suppose not. Consider the function $g(x) = f(x) - x$. Since f has no fixed point, $f(0) > 0$ so $g(0) > 0$. Also $f(1) < 1$ so that $g(1) < 0$. By the Intermediate Value Theorem, it follows that there is a point p at which $g(p) = 0$. But that means that $f(p) = p$. □