Math 4111
November 6, 2020 Lecture

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Figure: This is your instructor.

Definition: Let $E \subseteq \mathbb{R}$ be a set and let f be a real-valued function with domain E. Fix a point P which is in E and is also an accumulation point of E. We say that f is *continuous* at P if

$$\lim_{x \to P} f(x) = f(P).$$

We learned from the penultimate example of the last lecture that polynomial functions are continuous at every real x. So are the transcendental functions $\sin x$ and $\cos x$. A rational function is continuous at every point of its domain.

Example: The function

$$h(x) = \begin{cases} \sin(1/x) & \text{if} \quad x \neq 0 \\ 1 & \text{if} \quad x = 0 \end{cases}$$

is discontinuous at 0. See the next figure. The reason is that

$$\lim_{x \to 0} h(x)$$

does not exist. (Details of this assertion are left for you: notice that $h(1/(j\pi)) = 0$ while $h(2/[(4j+1)\pi]) = 1$ for $j = 1, 2, \ldots$.)

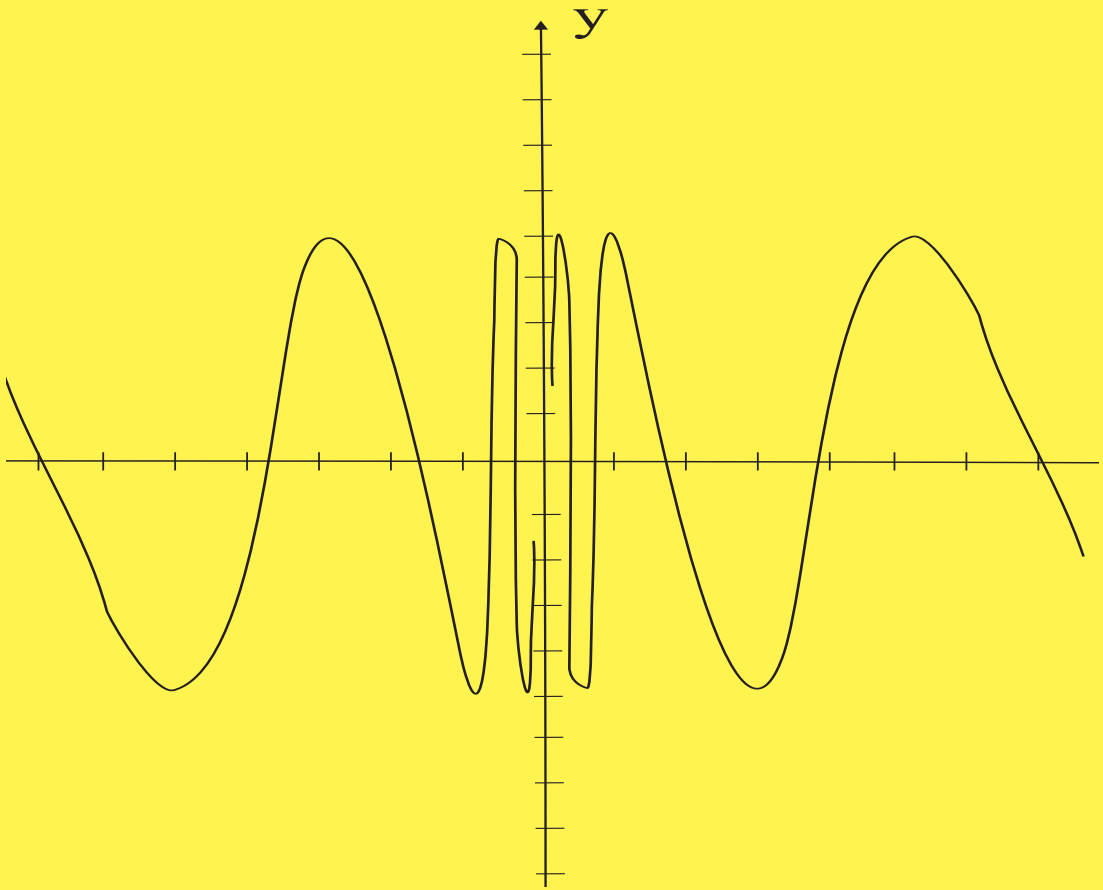


Figure: A function discontinuous at 0.

The function

$$k(x) = \begin{cases} x \cdot \sin(1/x) & \text{if} \quad x \neq 0 \\ 1 & \text{if} \quad x = 0 \end{cases}$$

is also discontinuous at $x = 0$. This time the limit $\lim_{x \to 0} k(x)$ exists, but the limit does not agree with $k(0)$.

However, the function

$$m(x) = \begin{cases} x \cdot \sin(1/x) & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad x = 0 \end{cases}$$

is continuous at $x = 0$ because the limit at 0 exists and agrees with the value of the function there. See the next figure. □

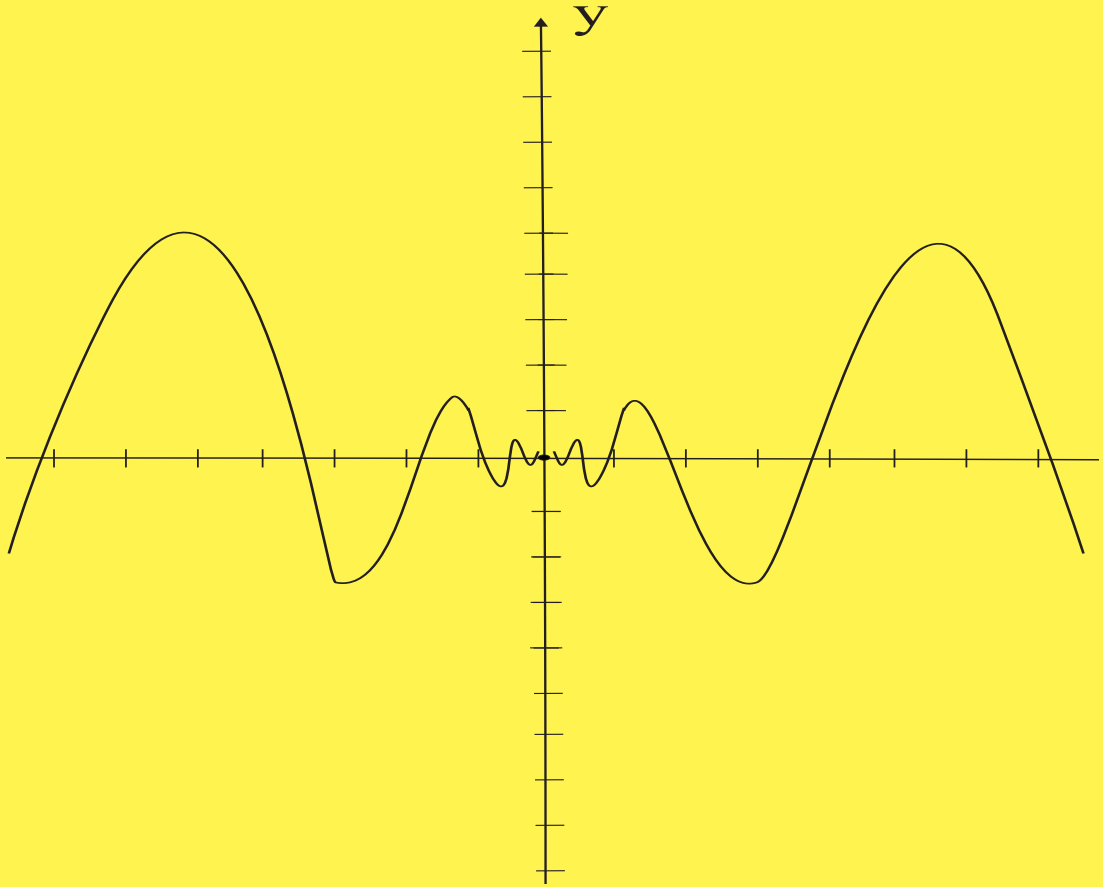


Figure: A function continuous at 0.

The arithmetic operations $+, -, \times$, and $\div$ preserve continuity (so long as we avoid division by zero). We now formulate this assertion as a theorem.

Theorem: *Let f and g be functions with domain E and let P be a point of E which is also an accumulation point of E. If f and g are continuous at P then so are $f \pm g$, $f \cdot g$, and (provided $g(P) \neq 0$) f/g.*

Proof: *Apply the relevant theorem from the last lecture.* □

Continuous functions may also be characterized using sequences:

Proposition: *Let f be a function with domain E and fix $P \in E$ which is an accumulation point of E. The function f is continuous at P if and only if, for every sequence $\{a_j\} \subseteq E$ satisfying $\lim_{j\to\infty} a_j = P$, it holds that*

$$\lim_{j\to\infty} f(a_j) = f(P).$$

Proof: Apply the appropriate proposition from the last lecture. □

Recall that, if g is a function with domain D and range E, and if f is a function with domain E and range F, then the *composition* of f and g is

$$f \circ g(x) = f(g(x)) .$$

See the figure.

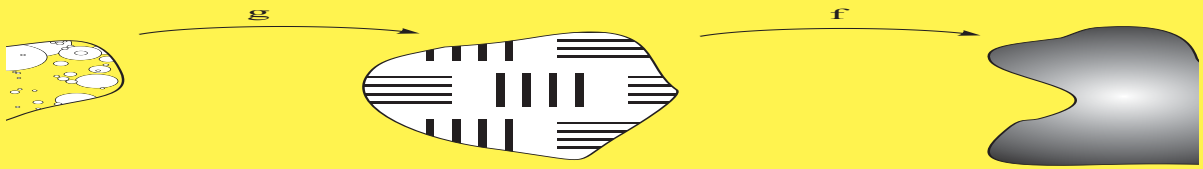


Figure: Composition of functions.

Proposition: *Let g have domain D and range E and let f have domain E and range F. Let $P \in D$. Suppose that P is an accumulation point of D and $g(P)$ is an accumulation point of E. Assume that g is continuous at P and that f is continuous at $g(P)$. Then $f \circ g$ is continuous at P.*

Proof: Let $\{a_j\}$ be any sequence in D such that $\lim_{j\to\infty} a_j = P$. Then

$$\lim_{j\to\infty} f \circ g(a_j) = \lim_{j\to\infty} f(g(a_j)) = f\left(\lim_{j\to\infty} g(a_j)\right)$$

$$= f\left(g\left(\lim_{j\to\infty} a_j\right)\right) = f(g(P)) = f \circ g(P)\,.$$

Now apply the last proposition. □

Example: It is not the case that if

$$\lim_{x \to P} g(x) = \ell$$

and

$$\lim_{t \to \ell} f(t) = m$$

then

$$\lim_{x \to P} f \circ g(x) = m\,.$$

A counterexample is given by the functions

$$g(x) = 0$$

$$f(x) = \begin{cases} 2 & \text{if} \quad x \neq 0 \\ 5 & \text{if} \quad x = 0. \end{cases}$$

Notice that $\lim_{x\to 0} g(x) = 0, \lim_{t\to 0} f(t) = 2$, yet $\lim_{x\to 0} f \circ g(x) = 5$.

The additional hypothesis that f be continuous at ℓ is necessary in order to guarantee that the limit of the composition will behave as expected. □

Next we explore the topological approach to the concept of continuity. Whereas the analytic approach that we have been discussing so far considers continuity one point at a time, the topological approach considers all points simultaneously. Let us call a function continuous if it is continuous at every point of its domain.

Definition: Let f be a function with domain E and let W be any set of real numbers. We define

$$f^{-1}(W) = \{x \in E : f(x) \in W\}.$$

We sometimes refer to $f^{-1}(W)$ as the *inverse image* of W under f.

Theorem: *Let f be a function with domain E. The function f is continuous if and only if the inverse image of any open set under f is the intersection of E with an open set.*

In particular, if E is open then f is continuous if and only if the inverse image of every open set under f is open.

Proof: Assume that f is continuous. Let $\mathcal{O}$ be any open set in $\mathbb{R}$ and let $P \in f^{-1}(\mathcal{O})$. Then, by definition, $f(P) \in \mathcal{O}$. Since $\mathcal{O}$ is open, there is an $\epsilon > 0$ such that the interval $(f(P) - \epsilon, f(P) + \epsilon)$ lies in $\mathcal{O}$. By the continuity of f we may select a $\delta > 0$ such that if $x \in E$ and $|x - P| < \delta$ then $|f(x) - f(P)| < \epsilon$. In other words, if $x \in E$ and $|x - P| < \delta$ then $f(x) \in \mathcal{O}$ or $x \in f^{-1}(\mathcal{O})$. Thus we have found an open interval $I = (P - \delta, P + \delta)$ about P whose intersection with E is contained in $f^{-1}(\mathcal{O})$. So $f^{-1}(\mathcal{O})$ is the intersection of E with an open set.

Conversely, suppose that for any open set $\mathcal{O} \subseteq \mathbb{R}$ we have that $f^{-1}(\mathcal{O})$ is the intersection of E with an open set. Fix $P \in E$. Choose $\epsilon > 0$. Then the interval $(f(P) - \epsilon, f(P) + \epsilon)$ is an open set. By hypothesis the set $f^{-1}((f(P) - \epsilon, f(P) + \epsilon))$ is the intersection of E with an open set.

This set contains the point P. Thus there is a $\delta > 0$ such that

$$E \cap (P - \delta, P + \delta) \subseteq f^{-1}((f(P) - \epsilon, f(P) + \epsilon)).$$

But that just says that

$$f(E \cap (P - \delta, P + \delta)) \subseteq (f(P) - \epsilon, f(P) + \epsilon).$$

In other words, if $|x - P| < \delta$ and $x \in E$ then $|f(x) - f(P)| < \epsilon$. But that means that f is continuous at P. □

Example: Since any open subset of the real numbers is a countable or finite disjoint union of intervals then—in order to check that the inverse image under a function f of every open set is open—it is enough to check that the inverse image of any open interval is open. This is frequently easy to do.

For example, if $f(x) = x^2$ then the inverse image of an open interval (a, b) is $(-\sqrt{b}, -\sqrt{a}) \cup (\sqrt{a}, \sqrt{b})$ if $a > 0$, is $(-\sqrt{b}, \sqrt{b})$ if $a \leq 0, b \geq 0$, and is $\emptyset$ if $a < b < 0$. Thus the function f is continuous.

Note that, by contrast, it is somewhat tedious to give an $\epsilon - \delta$ proof of the continuity of $f(x) = x^2$. □

Corollary: *Let f be a function with domain E. The function f is continuous if and only if the inverse image of any closed set F under f is the intersection of E with some closed set.*

In particular, if E is closed then f is continuous if and only if the inverse image of any closed set F under f is closed.

Proof: It is enough to prove that

$$f^{-1}\left({}^{c}F\right) = {}^{c}\left(f^{-1}(F)\right) .$$

We leave this assertion as an exercise for you. □