Math 4111
November 4, 2020 Lecture

Steven G. Krantz



Figure: This is your instructor.

In this chapter we are going to treat some topics that you have seen before in your calculus class. However, we shall use the deep properties of the real numbers that we have developed in this text to obtain important new insights. Therefore you should *not* think of this chapter as review. Look at the concepts introduced here with the power of your new understanding of analysis.

Definition: Let $E \subseteq \mathbb{R}$ be a set and let f be a real-valued function with domain E. Fix a point $P \in \mathbb{R}$ that is an accumulation point of E. Let ℓ be a real number. We say that

$$\lim_{E \ni x \to P} f(x) = \ell$$

if, for each $\epsilon > 0$, there is a $\delta > 0$ such that, when $x \in E$ and $0 < |x - P| < \delta$, then

$$|f(x) - \ell| < \epsilon \, .$$

In words, we say that the limit as x tends to P of f is equal to ℓ.

The definition makes precise the notion that we can force $f(x)$ to be just as close as we please to ℓ by making x sufficiently close to P. Notice that the definition puts the condition $0 < |x - P| < \delta$ on x, so that x is not allowed to take the value P. In other words we do not look at $x = P$, but rather at x *near* to P.

Also observe that we only consider the limit of f at a point P that is not isolated. In the exercises you will be asked to discuss why it would be nonsensical to use the above definition to study the limit at an isolated point.

Example: Let $E = \mathbb{R} \setminus \{0\}$ and

$$f(x) = x \cdot \sin(1/x) \quad \text{if} \quad x \in E\,.$$

See the figure. Then $\lim_{x\to 0} f(x) = 0$. To see this, let $\epsilon > 0$. Choose $\delta = \epsilon$. If $0 < |x - 0| < \delta$ then

$$|f(x) - 0| = |x \cdot \sin(1/x)| \leq |x| < \delta = \epsilon\,,$$

as desired. Thus the limit exists and equals 0. □

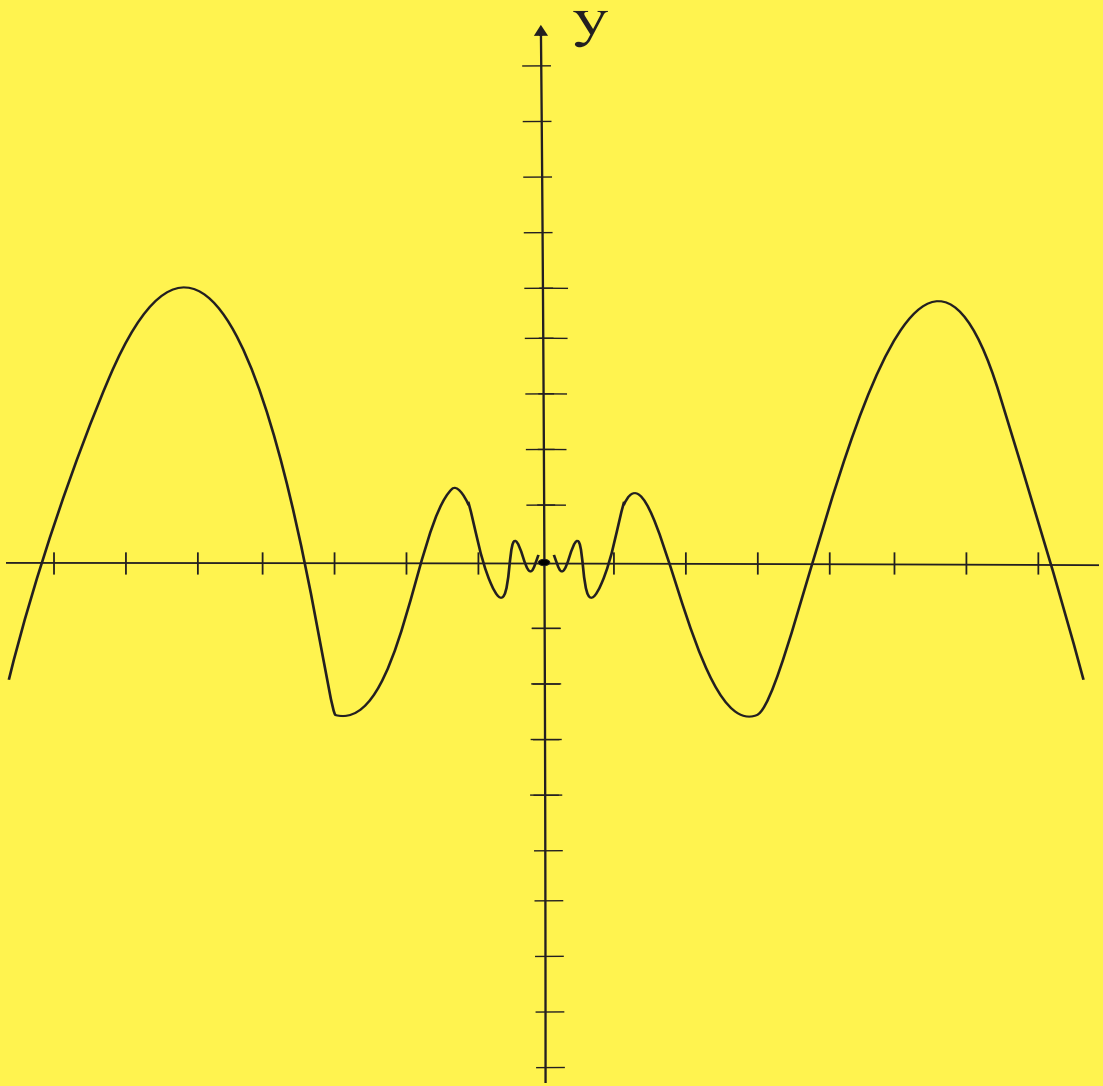


Figure: The limit of an oscillatory function.

Example: Let $E = \mathbb{R}$ and

$$g(x) = \begin{cases} 1 & \text{if} \quad x \text{ is rational} \\ 0 & \text{if} \quad x \text{ is irrational.} \end{cases}$$

Then $\lim_{x \to P} g(x)$ does not exist for any point P of E.

To see this, fix $P \in \mathbb{R}$. Seeking a contradiction, assume that there is a limiting value ℓ for g at P. If this is so, then we take $\epsilon = 1/2$ and we can find a $\delta > 0$ such that $0 < |x - P| < \delta$ implies

$$|g(x) - \ell| < \epsilon = \frac{1}{2}\,.$$

If we take x to be rational then the last line says that

$$|1 - \ell| < \frac{1}{2}\,,$$

while if we take x irrational then the first line says that

$$|0 - \ell| < \frac{1}{2}\,.$$

But then the triangle inequality gives that

$$\begin{aligned} |1-0| &= |(1-\ell)+(\ell-0)| \\ &\leq |1-\ell|+|\ell-0|, \end{aligned}$$

which by the last two lines is

$$< 1\,.$$

This contradiction, that $1 < 1$, allows us to conclude that the limit does not exist at P. $\square$

Proposition: *Let f be a function with domain E, and let P be an accumulation point of E. If $\lim_{x \to P} f(x) = \ell$ and $\lim_{x \to P} f(x) = m$, then $\ell = m$.*

Proof: Let $\epsilon > 0$. Choose $\delta_1 > 0$ such that, if $x \in E$ and $0 < |x - P| < \delta_1$, then $|f(x) - \ell| < \epsilon/2$. Similarly choose $\delta_2 > 0$ such that, if $x \in E$ and $0 < |x - P| < \delta_2$, then $|f(x) - m| < \epsilon/2$. Define δ to be the minimum of δ_1 and δ_2. If $x \in E$ and $0 < |x - P| < \delta$, then the triangle inequality tells us that

$$\begin{aligned} |\ell - m| &= |(\ell - f(x)) + (f(x) - m)| \\ &\leq |(\ell - f(x)| + |f(x) - m)| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon \end{aligned}$$

Since $|\ell - m| < \epsilon$ for every positive ϵ we conclude that $\ell = m$. That is the desired result. $\square$

The point of the last proposition is that if a limit is calculated by two different methods, then the same answer will result. While of primarily philosophical interest now, this will be important information later when we establish the existence of certain limits.

This is a good time to observe that the limits

$$\lim_{x \to P} f(x)$$

and

$$\lim_{h \to 0} f(P + h)$$

are equal in the sense that if one limit exists then so does the other and they both have the same value.

In order to facilitate checking that certain limits exist, we now record some elementary properties of the limit. This requires that we first recall how functions are combined.

Suppose that f and g are each functions which have domain E. We define the *sum* or *difference* of f and g to be the function

$$(f \pm g)(x) = f(x) \pm g(x),$$

the *product* of f and g to be the function

$$(f \cdot g)(x) = f(x) \cdot g(x),$$

and the quotient of f and g to be

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}.$$

Notice that the quotient is only defined at points x for which $g(x) \neq 0$. Now we have:

Theorem: [Elementary Properties of Limits of Functions] *Let f and g be functions with domain E and fix a point P that is an accumulation point of E. Assume that*

(i) $\lim_{x \to P} f(x) = \ell$

(ii) $\lim_{x \to P} g(x) = m$.

Then

(a) $\lim_{x \to P} (f \pm g)(x) = \ell \pm m$

(b) $\lim_{x \to P} (f \cdot g)(x) = \ell \cdot m$

(c) $\lim_{x \to P} (f/g)(x) = \ell/m$ *provided* $m \neq 0$.

Proof: We prove part **(b)**. Parts **(a)** and **(c)** are treated in the exercises.

Let $\epsilon > 0$. We may also assume that $\epsilon < 1$. Choose $\delta_1 > 0$ such that, if $x \in E$ and $0 < |x - P| < \delta_1$, then

$$|f(x) - \ell| < \frac{\epsilon}{2(|m| + 1)}.$$

Choose $\delta_2 > 0$ such that if $x \in E$ and $0 < |x - P| < \delta_2$ then

$$|g(x) - m| < \frac{\epsilon}{2(|\ell| + 1)}.$$

(Notice that this last inequality implies that $|g(x)| < |m| + |\epsilon|$.)

Let δ be the minimum of δ_1 and δ_2. If $x \in E$ and $0 < |x - P| < \delta$ then

$$\begin{aligned}
|f(x) \cdot g(x) - \ell \cdot m| &= |(f(x) - \ell) \cdot g(x) + (g(x) - m) \cdot \ell| \\
&\leq |(f(x) - \ell) \cdot g(x)| + |(g(x) - m) \cdot \ell| \\
&< \left(\frac{\epsilon}{2(|m| + 1)} \right) \cdot |g(x)| + \left(\frac{\epsilon}{2(|\ell| + 1)} \right) \cdot |\ell| \\
&\leq \left(\frac{\epsilon}{2(|m| + 1)} \right) \cdot (|m| + |\epsilon|) + \frac{\epsilon}{2} \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon \,.
\end{aligned}$$

□

Thus the limit exists and equals $\ell \cdot m$.

Example: It is a simple matter to check that, if $f(x) = x$, then

$$\lim_{x \to P} f(x) = P$$

for every real P. (Indeed, for $\epsilon > 0$ we may take $\delta = \epsilon$.) Also, if $g(x) \equiv \alpha$ is the constant function taking value α, then

$$\lim_{x \to P} g(x) = \alpha \, .$$

It then follows from parts **(a)** and **(b)** of the theorem that, if $f(x)$ is any polynomial function, then

$$\lim_{x \to P} f(x) = f(P)\,.$$

Moreover, if $r(x)$ is any *rational function* (quotient of polynomials) then we may also use part **(c)** of the theorem to conclude that

$$\lim_{x \to P} r(x) = r(P)$$

for all points P at which the rational function $r(x)$ is defined. $\square$

Example: If x is a small, positive real number then $0 < \sin x < x$. This is true because $\sin x$ is the nearest distance from the point $(\cos x, \sin x)$ to the x-axis while x is the distance from that point to the x-axis along an arc. If $\epsilon > 0$, then we set $\delta = \epsilon$. We conclude that if $0 < |x - 0| < \delta$ then

$$|\sin x - 0| < |x| < \delta = \epsilon\,.$$

Since $\sin(-x) = -\sin x$, the same result holds when x is a negative number with small absolute value. Therefore

$$\lim_{x \to 0} \sin x = 0\,.$$

Since

$$\cos x = \sqrt{1 - \sin^2 x} \quad \text{for all} \quad x \in [-\pi/2, \pi/2]\,,$$

we may conclude from the preceding theorem that

$$\lim_{x \to 0} \cos x = 1\,.$$

Now fix any real number P. We have

$$\begin{aligned}
\lim_{x \to P} \sin x &= \lim_{h \to 0} \sin(P + h) \\
&= \lim_{h \to 0} \big(\sin P \cos h + \cos P \sin h\big) \\
&= \sin P \cdot 1 + \cos P \cdot 0 \\
&= \sin P \,.
\end{aligned}$$

We of course have used parts **(a)** and **(b)** of the theorem to commute the limit process with addition and multiplication. A similar argument shows that

$$\lim_{x \to P} \cos x = \cos P \,. \qquad \square$$

Remark: In the last example, we have used the definition of the sine function and the cosine function that you learned in calculus. In a later lecture, when we learn about series of functions, we will learn a more rigorous method for treating the trigonometric functions.

We conclude by giving a characterization of the limit of a function using sequences.

Proposition: *Let f be a function with domain E and P be an accumulation point of E. Then*

$$\lim_{x \to P} f(x) = \ell$$

if and only if, for any sequence $\{a_j\} \subseteq E \setminus \{P\}$ satisfying $\lim_{j\to\infty} a_j = P$, it holds that

$$\lim_{j \to \infty} f(a_j) = \ell\,.$$

Proof: Assume that the first condition fails. Then there is an $\epsilon > 0$ such that for no $\delta > 0$ is it the case that when $0 < |x - P| < \delta$ then $|f(x) - \ell| < \epsilon$. Thus, for each $\delta = 1/j$, we may choose a number $a_j \in E \setminus \{P\}$ with $0 < |a_j - P| < 1/j$ and $|f(a_j) - \ell| \geq \epsilon$. But then the second condition fails for this sequence $\{a_j\}$.

If the second condition fails then there is some sequence $\{a_j\}$ such that $\lim_{j\to\infty} a_j = P$ but $\lim_{j\to\infty} f(a_j) \neq \ell$. This means that there is an $\epsilon > 0$ such that for infinitely many a_j it holds that $|f(a_j) - \ell| \geq \epsilon$. But then, no matter how small $\delta > 0$, there will be an a_j satisfying $0 < |a_j - P| < \delta$ (since $a_j \to P$) and $|f(a_j) - \ell| \geq \epsilon$. Thus the first condition fails. $\square$