Math 4111
November 2, 2020 Lecture

Steven G. Krantz



Figure: This is your instructor.

Today we will have a short lesson on perfect sets. Anyway, this is a good way to end the chapter.

Definition: A set $S \subseteq \mathbb{R}$ is called *perfect* if it is closed and if every point of S is an accumulation point of S.

The property of being perfect is a rather special one: it means that the set has no isolated points.

Example: Consider the set $S = [0, 2]$. This set is perfect. Because **(i)** it is closed, **(ii)** any interior point is clearly an accumulation point, **(iii)** 0 is the limit of $\{1/j\}$ so is an accumulation point, and **(iv)** 2 is the limit of $\{2 - 1/j\}$ so is an accumulation point. □

Clearly any closed interval $[a, b]$ is perfect. After all, a point x in the interior of the interval is surrounded by an entire open interval $(x - \epsilon, x + \epsilon)$ of elements of the interval; moreover a is the limit of elements from the right and b is the limit of elements from the left.

Example: The Cantor set, *a totally disconnected set*, is perfect. It is definitely closed. Now fix $x \in C$. Then $x \in S_1$. Thus x is in one of the two intervals composing S_1. One (or perhaps both) of the endpoints of that interval does not equal x. Call that endpoint a_1. Likewise $x \in S_2$. Therefore x lies in one of the intervals of S_2. Choose an endpoint a_2 of that interval which does not equal x. Continuing in this fashion, we construct a sequence $\{a_j\}$. Notice that *each of the elements of this sequence lies in the Cantor set* (why?). Finally, $|x - a_j| \leq 3^{-j}$ for each j. Therefore x is the limit of the sequence. We have thus proved that the Cantor set is perfect. □

The fundamental theorem about perfect sets tells us that such a set must be rather large. We have

Theorem: *A nonempty perfect set must be uncountable.*

Proof: Let S be a nonempty perfect set. Since S has accumulation points, it cannot be finite. Therefore it is either countable or uncountable.

Seeking a contradiction, we suppose that S is countable. Write $S = \{s_1, s_2, \dots\}$. Set $U_1 = (s_1 - 1, s_1 + 1)$. Then U_1 is a neighborhood of s_1. Now s_1 is a limit point of S so there must be infinitely many elements of S lying in U_1. We select a bounded open interval U_2 such that $\overline{U}_2 \subseteq U_1$, $\overline{U}_2$ does not contain s_1, and U_2 *does* contain some element of S.

Continuing in this fashion, assume that $s_1, \ldots, s_j$ have been selected and choose a bounded interval U_{j+1} such that (i) $\overline{U}_{j+1} \subseteq U_j$, (ii) $s_j \notin \overline{U}_{j+1}$, and (iii) U_{j+1} contains some element of S.

Observe that each set $V_j = \overline{U}_j \cap S$ is closed and bounded, hence compact. Also each V_j is nonempty by construction but V_j does not contain s_{j-1}. It follows that $V = \cap_j V_j$ cannot contain s_1 (since V_2 does not), cannot contain s_2 (since V_3 does not), indeed cannot contain any element of S. Hence V, being a subset of S, is empty. But V is the decreasing intersection of nonempty compact sets, hence cannot be empty!

This contradiction shows that S cannot be countable. So it must be uncountable. □

Corollary: *If $a < b$ then the closed interval $[a, b]$ is uncountable.*
Proof: *The interval $[a, b]$ is perfect.* □

We also have a new way of seeing that the Cantor set is uncountable, since it is perfect:

Corollary: *The Cantor set is uncountable.*