Math 4111
October 30, 2020 Lecture

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Figure: This is your instructor.

4.5 Connected and Disconnected Sets

Definition

Let S be a set of real numbers. We say that S is *disconnected* if it is possible to find a pair of open sets U and V such that

(1) $U \cap S \neq \emptyset, V \cap S \neq \emptyset$,

(2) $(U \cap S) \cap (V \cap S) = \emptyset$, and

(3) $S = (U \cap S) \cup (V \cap S)$.

See the figure below. If no such U and V exist, then we say that S is *connected*.

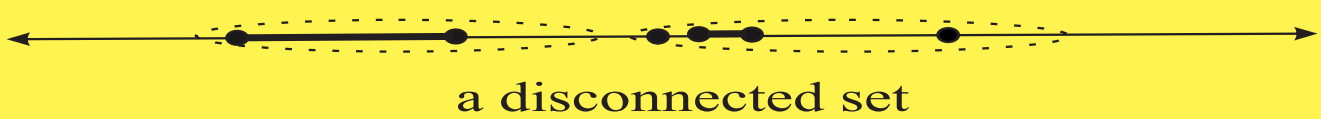


Figure: The idea of disconnected.

Note that in the above definition, condition (1) means that each of U and V contains some element of S, while condition (2) says that no common element of U and V is in S. Condition (3) means that $S \subseteq U \cup V$.

Example

The set $T = \{x \in \mathbb{R} : |x| < 1, x \neq 0\}$ is disconnected. Take $U = \{x : x < 0\}$ and $V = \{x : x > 0\}$. Then

$U \cap T = \{x : -1 < x < 0\} \neq \emptyset$ and $V \cap T = \{x : 0 < x < 1\} \neq \emptyset$.

Also $(U \cap T) \cap (V \cap T) = \emptyset$. Clearly $T = (U \cap T) \cup (V \cap T)$. Hence, T is disconnected. □

It is clear that a disconnected set has the property that there are disjoint open sets that, in effect, *disconnect* the set. The next example looks at the contrapositive.

Example

The set $X = [-1, 1]$ is connected. To see this, suppose to the contrary that there exist open sets U and V such that $U \cap X \neq \emptyset, V \cap X \neq \emptyset, (U \cap X) \cap (V \cap X) = \emptyset$, and

$$X = (U \cap X) \cup (V \cap X) \ .$$

Choose $a \in U \cap X$ and $b \in V \cap X$. We may assume $a < b$. Set

$$\alpha = \sup \left(U \cap [a, b] \} \right) \ .$$

Now $[a, b] \subseteq X$ hence $U \cap [a, b]$ is disjoint from V. Thus $\alpha \leq b$. But ${}^c V$ is closed hence $\alpha \notin V$. It follows that $\alpha < b$.

If $\alpha \in U$ then, because U is open, there exists an $\widetilde{\alpha} \in U$ such that $\alpha < \widetilde{\alpha} < b$. This would mean that we chose α incorrectly. Hence $\alpha \notin U$. But $\alpha \notin U$ and $\alpha \notin V$ means $\alpha \notin X$. On the other hand, α is the supremum of a subset of X (since $a \in X, b \in X$, and X is an interval). Since X is a closed interval, we conclude that $\alpha \in X$. This contradiction shows that X must be connected. $\square$

With small modifications, the discussion in the last example demonstrates that any closed interval is connected (Exercise **1**). See figure below. Also (see Exercise **2**), we may similarly see that any open interval or half-open interval is connected. In fact the converse is true as well.



Figure: A closed interval is connected.

Theorem

A subset S of $\mathbb{R}$ is connected if and only if S is an interval.

Proof: If S is not an interval then there exist $a \in S, b \in S$ and a point t between a and b such that $t \notin S$. Define $U = \{x \in \mathbb{R} : x < t\}$ and $V = \{x \in \mathbb{R} : t < x\}$. Then U and V are open and disjoint, $U \cap S \neq \emptyset$, $V \cap S \neq \emptyset$, and

$$S = (U \cap S) \cup (V \cap S) \, .$$

Thus S is disconnected.

If S is an interval then we prove that it is connected using the methodology of Example 4.45. $\square$

The Cantor set is not connected; indeed it is disconnected in a special sense. Call a set S *totally disconnected* if, for each distinct $x \in S$, $y \in S$, there exist disjoint open sets U and V such that $x \in U, y \in V$, and $S = (U \cap S) \cup (V \cap S)$.

Proposition

The Cantor set is totally disconnected.

Proof: Let $x, y \in C$ be distinct and assume that $x < y$. Set $\delta = |x - y|$. Choose j so large that $3^{-j} < \delta$. Then $x, y \in S_j$, but x and y cannot both be in the same interval of S_j (since the intervals will have length equal to 3^{-j}). It follows that there is a point t between x and y that is not an element of S_j, hence not an element of C. Set $U = \{s : s < t\}$ and $V = \{s : s > t\}$. Then $x \in U \cap C$ hence $U \cap C \neq \emptyset$; likewise, $y \in V \cap C \neq \emptyset$. Also $(U \cap C) \cap (V \cap C) = \emptyset$. Finally $C = (C \cap U) \cup (C \cap V)$. Thus C is totally disconnected. $\square$