Math 4111
October 26, 2020 Lecture

Steven G. Krantz



Figure: This is your instructor.

4.3 Compact Sets

Exercises: 1–10.

Compact sets are sets (usually infinite) which share many of the most important properties of finite sets. They play an important role in real analysis.

Definition
A set $S \subseteq \mathbb{R}$ is called *compact* if every sequence in S has a subsequence that converges *to an element of* S.

Theorem (Heine–Borel)

A set $S \subseteq \mathbb{R}$ is compact if and only if it is closed and bounded.

Proof: That a closed, bounded set has the property of compactness is the content of Corollary 4.29 and Proposition 4.13.

Now let S be a set that is compact. If S is not bounded, then there is an element s_1 of S that has absolute value larger than 1. Also there must be an element s_2 of S that has absolute value larger than 2. Continuing, we find elements $s_j \in S$ satisfying

$$|s_j| > j$$

for each j. But then no subsequence of the sequence $\{s_j\}$ can be Cauchy. This contradiction shows that S must be bounded.

If S is compact but S is not closed, then there is a point x which is the limit of a sequence $\{s_j\} \subseteq S$ but which is not itself in S. But every sequence in S is, by definition of "compact," supposed to have a subsequence converging *to an element of* S. For the sequence $\{s_j\}$ that we are considering, x is the only candidate for the limit of a subsequence. Thus it must be that $x \in S$. That contradiction establishes that S is closed. $\square$

It is worth noting that Heine-Borel Theorem holds in $\mathbb{R}^n$ for all n, but it does not hold in more general metric spaces (such as infinite dimensional inner product spaces).

In the abstract theory of topology (where there is no notion of distance), sequences are not very useful for describing topological properties. Therefore a different definition of compactness is used.

For interest's sake, and for future use, we now show that the definition of compactness that we have been discussing is equivalent to the one used in abstract topology theory. First we need a new definition.

Definition

Let $S \subseteq \mathbb{R}$. A collection of open sets $\{\mathcal{O}_\alpha\}_{\alpha \in A}$ (where each $\mathcal{O}_\alpha$ is an open set of real numbers) is called an *open covering* (or an open cover) of S if

$$S \subseteq \bigcup_{\alpha \in A} \mathcal{O}_\alpha.$$

See the figure below.



an open cover of the set S

It is the distinction displayed in this example that distinguishes compact sets from the point of view of topology. To understand the point, we need another definition:

Definition

If $\mathcal{C}$ is an open covering of a set S and if $\mathcal{D}$ is another open covering of S such that each element of $\mathcal{D}$ is also an element of $\mathcal{C}$ then we call $\mathcal{D}$ a *subcovering* of $\mathcal{C}$.

We call $\mathcal{D}$ a *finite subcovering* if $\mathcal{D}$ has just finitely many elements.

Example

The collection of intervals

$$\mathcal{C} = \{(j-1, j+1)\}_{j=1}^{\infty}$$

is an open covering of the set $S = [5, 9]$. The collection

$$\mathcal{D} = \{(j-1, j+1)\}_{j=5}^{\infty}$$

is a subcovering.

However, the collection

$$\mathcal{E} = \{(4, 6), (5, 7), (6, 8), (7, 9), (8, 10)\}$$

is a *finite* subcovering. □

Theorem

A set $S \subseteq \mathbb{R}$ is compact if and only if every open covering $\mathcal{C} = \{\mathcal{O}_\alpha\}_{\alpha \in A}$ of S has a finite subcovering.

Proof: Assume that S is a compact set and let $\mathcal{C} = \{\mathcal{O}_\alpha\}_{\alpha \in A}$ be an open covering of S.

We may assume that S is nonempty. By Heine-Borel Theorem , S is closed and bounded. Therefore it holds that $a = \inf S$ is a finite real number, and an element of S. Likewise, $b = \sup S$ is a finite real number and an element of S. Write $I = [a, b]$. The case $a = b$ is trivial so we assume that $a < b$.

Set

$$\mathcal{A} = \{x \in I : \mathcal{C} \text{ contains a finite subcover that covers } S \cap [a, x]\}.$$

Then $\mathcal{A}$ is nonempty since $a \in \mathcal{A}$. Let $t = \sup \mathcal{A}$. Then some element $\mathcal{O}_0$ of $\mathcal{C}$ contains t. Let s be an element of $\mathcal{O}_0$ to the left of t. Then, by the definition of t, s is an element of $\mathcal{A}$. So there is a finite subcovering $\mathcal{C}'$ of $\mathcal{C}$ that covers $S \cap [a, s]$. But then $\mathcal{D} = \mathcal{C}' \cup \{\mathcal{O}_0\}$ covers $S \cap [a, t]$, showing that $t = \sup \mathcal{A}$ lies in A. But in fact $\mathcal{D}$ even covers points to the right of t. Thus t cannot be the supremum of $\mathcal{A}$ unless $t = b$.

We have learned that t must be the point b itself and that therefore $b \in \mathcal{A}$. But that says that $S \cap [a, b] = S$ can be covered by finitely many of the elements of $\mathcal{C}$. That is what we wished to prove.

For the converse, assume that every open covering of S has a finite subcovering. Let $\{a_j\}$ be a sequence in S. Assume, seeking a contradiction, that the sequence has no subsequence that converges to an element of S. This means that for every $s \in S$ there is an $\epsilon_s > 0$ such that the interval $I_s = (s - \epsilon_s, s + \epsilon_s)$ contains at most finitely many terms of the sequence $\{a_j\}$. The collection $\mathcal{C} = \{I_s\}$ is then an open covering of the set S. By hypothesis, there exists a finite subcovering $I_{s_1}, \ldots I_{s_k}$ of open intervals that cover S. But then $S \subseteq \cup_{j=1}^{k} I_{s_j}$ contains only finitely many terms of the sequence $\{a_j\}$, which is a contradiction. $\square$

Example

If $A \subseteq B$ and both sets are nonempty then $A \cap B = A \neq \emptyset$. A similar assertion holds when intersecting *finitely many* nonempty sets $A_1 \supseteq A_2 \supseteq \cdots \supseteq A_k$; it holds in this circumstance that $\cap_{j=1}^{k} A_j = A_k$.

However, it is possible to have infinitely many nonempty nested sets with null intersection. An example is the sets $I_j = (0, 1/j)$. We see that $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$ yet $\bigcap_{j=1}^{\infty} I_j = \emptyset$.

By contrast, if we take $K_j = [0, 1/j]$ then

$$\bigcap_{j=1}^{\infty} K_j = \{0\}.$$

The next proposition shows that compact sets have the intuitively appealing property of the K_js rather than the unsettling property of the I_js. □

Proposition

Let

$$K_1 \supseteq K_2 \supseteq \cdots \supseteq K_j \supseteq \ldots$$

be nonempty compact sets of real numbers. Set

$$\mathcal{K} = \bigcap_{j=1}^{\infty} K_j \, .$$

Then $\mathcal{K}$ is compact and $\mathcal{K} \neq \emptyset$.

Proof: Each K_j is closed and bounded hence $\mathcal{K}$ is closed and bounded. Thus $\mathcal{K}$ is compact. Let $x_j \in K_j$, each j. Then $\{x_j\} \subseteq K_1$. By compactness, there is a convergent subsequence $\{x_{j_k}\}$ with limit $x_0 \in K_1$. However, $\{x_{j_k}\}_{k=2}^{\infty} \subseteq K_2$. Thus $x_0 \in K_2$. Similar reasoning shows that $x_0 \in K_m$ for all $m = 1, 2, \ldots$. In conclusion, $x_0 \in \cap_j K_j = \mathcal{K}$. □

Corollary

Let $[a_1, b_1] \supseteq [a_2, b_2] \supseteq \cdots \supseteq [a_n, b_n] \supseteq \cdots$ *be nested closed intervals in* $\mathbb{R}$ *such that* $\lim_{n\to\infty}(b_n - a_n) = 0$. *Then there is a unique number* c *such that* $c \in [a_n, b_n]$ *for all* n.

For two nonempty subsets A and B of $\mathbb{R}$, the *distance* between A and B, denoted by $\operatorname{dis}(A, B)$, is defined as

$$\operatorname{dis}(A, B) = \inf\{|x - y| : x \in A, \text{ and } y \in B\}.$$

Note that two disjoint closed sets may have distance 0. For instance, consider $A = \mathbb{N}$ and $B = \{n + \frac{1}{n+1} : n \in \mathbb{N}\}$.

We conclude with a property of the distance between a compact set and a closed set.

Exercise

Let K and L be nonempty, disjoint subsets of $\mathbb{R}$. Assume that K is compact and L is closed. Show that there exist $x \in K$ and $y \in L$ such that $|x - y| = \text{dis}(K, L)$.

Hints: Let $d = \text{dis}(K, L)$. For each $n \in \mathbb{N}$, take $x_n \in K$ and $y_n \in L$ with $\big||x_n - y_n| - d\big| < 1/n$. Since K is compact, (x_n) has a subsequece (x_{n_k}) that converges to some $a \in K$. Note that (y_{n_k}) is bounded and hence has a subsequence $(y_{n_{k_m}})$ that converges to some $b \in L$. Explain why $d = |a - b|$.