Math 4111
October 23, 2020 Lecture

Steven G. Krantz

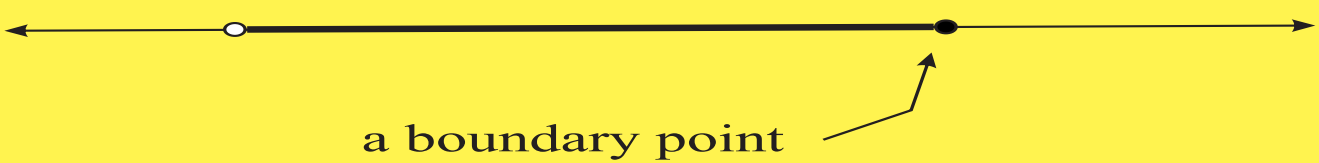


Figure: This is your instructor.

Further Properties of Open and Closed Sets

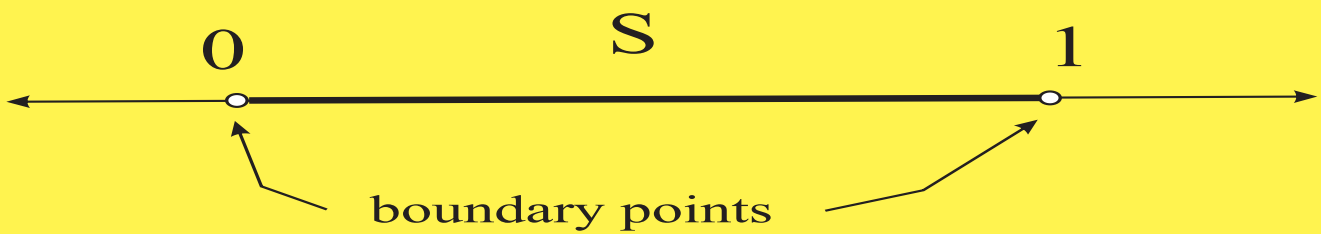
Definition

Let $S \subseteq \mathbb{R}$ be a set. We call $b \in \mathbb{R}$ a *boundary point* of S if every nonempty neighborhood $(b - \epsilon, b + \epsilon)$ contains both points of S and points of $\mathbb{R} \setminus S$. See the figure. We denote the set of boundary points of S by ∂S.


a boundary point

A boundary point b might lie in S and might lie in the complement of S. The next example serves to illustrate the concept:

Example: Let S be the interval $(0, 1)$. Then no point of $(0, 1)$ is in the boundary of S since every point of $(0, 1)$ has a neighborhood that lies entirely inside $(0, 1)$. Also, no point of the complement of $T = [0, 1]$ lies in the boundary of S for a similar reason. Indeed, the only candidates for elements of the boundary of S are 0 and 1. See the figure. The point 0 *is* an element of the boundary since every neighborhood $(0 - \epsilon, 0 + \epsilon)$ contains the point $\epsilon/2 \in S$ and the point $-\epsilon/2 \in \mathbb{R} \setminus S$. A similar calculation shows that 1 lies in the boundary of S.


0
S
1
boundary points

Now consider the set $T = [0, 1]$. Certainly there are no boundary points in $(0, 1)$, for the same reason as in the first paragraph. And there are no boundary points in $\mathbb{R} \setminus [0, 1]$, since that set is open. Thus the only candidates for elements of the boundary are 0 and 1. As in the first paragraph, these are both indeed boundary points for T. See the figure.


0
T
1
boundary points

Notice that neither of the boundary points of S lie in S while both of the boundary points of T lie in T. □

The collection of all boundary points of a set S is called the *boundary* of S and is denoted by ∂S.

Example: The boundary of the set $\mathbb{Q}$ is the entire real line. For if x is any element of $\mathbb{R}$ then every interval $(x - \epsilon, x + \epsilon)$ contains both rational numbers and irrational numbers. $\square$

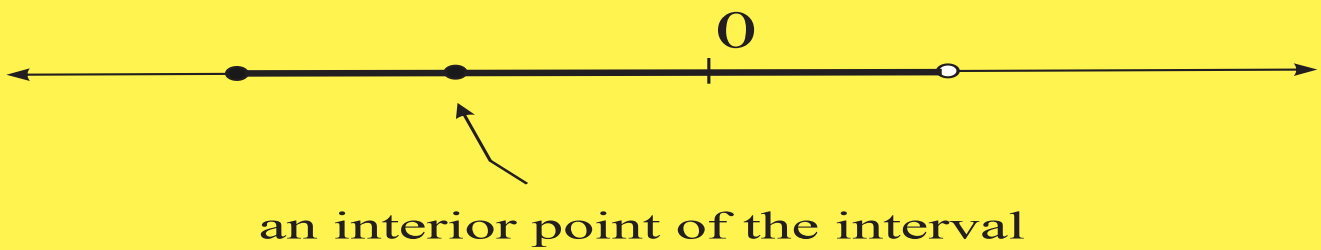
The union of a set S with its boundary is called the *closure* of S, denoted $\overline{S}$. The next example illustrates the concept.

Example: Let S be the set of rational numbers in the interval $[0, 1]$. Then the closure $\overline{S}$ of S is the entire interval $[0, 1]$.

Let T be the open interval $(0, 1)$. Then the closure $\overline{T}$ of T is the closed interval $[0, 1]$. □

Definition

Let $S \subseteq \mathbb{R}$. A point $s \in S$ is called an *interior point* of S if there is an $\epsilon > 0$ such that the interval $(s - \epsilon, s + \epsilon)$ lies in S. See the figure. We call the set of all interior points the *interior* of S, and we denote this set by $\overset{\circ}{S}$.


0
an interior point of the interval

Definition

A point $t \in S$ is called an *isolated point* of S if there is an $\epsilon > 0$ such that the intersection of the interval $(t - \epsilon, t + \epsilon)$ with S is just the singleton $\{t\}$. See the figure.


isolated points

By the definitions given here, an isolated point t of a set $S \subseteq \mathbb{R}$ is a boundary point. For any interval $(t-\epsilon, t+\epsilon)$ contains a point of S (namely, t itself) and points of $\mathbb{R} \setminus S$ (since t is isolated).

Proposition: *Let $S \subseteq \mathbb{R}$. Then each point of S is either an interior point or a boundary point of S.*

Proof: Fix $s \in S$. If s is not an interior point then no open interval centered at s contains only elements of s. Thus any interval centered at s contains an element of S (namely, s itself) and also contains points of $\mathbb{R} \setminus S$. Thus s is a boundary point of S. $\square$

Example: Let $S = [0, 1]$. Then the interior points of S are the elements of $(0, 1)$. The boundary points of S are the points 0 and 1. The set S has no isolated points.

Let $T = \{1, 1/2, 1/3, \dots\} \cup \{0\}$. Then the points 1, 1/2, 1/3, ... are isolated points of T. The point 0 is an accumulation point of T. Every element of T is a boundary point, and there are no others. □

Remark: Observe that the interior points of a set S are *elements* of S—by their very definition. Also isolated points of S are elements of S. However, a boundary point of S may or may not be an element of S.

If x is an accumulation point of S then every open neighborhood of x contains infinitely many elements of S. Hence x is either a boundary point of S or an interior point of S; it *cannot* be an isolated point of S.

Proposition: *Let S be a subset of the real numbers. Then the boundary of S equals the boundary of $\mathbb{R} \setminus S$.*

Proof: *If x is in the boundary of S, then any neighborhood of x contains points of S and points of ${}^cS$. Thus every neighborhood of x contains points of ${}^cS$ and points of S. So x is in the boundary of ${}^cS$.* □

The next theorem allows us to use the concept of boundary to distinguish open sets from closed sets.

Theorem

A closed set contains all of its boundary points. An open set contains none of its boundary points.

Proof: Let S be closed and let x be an element of its boundary. If every neighborhood of x contains points of S *other than* x *itself* then x is an accumulation point of S hence $x \in S$. If not every neighborhood of x contains points of S other than x itself, then there is an $\epsilon > 0$ such that $\{(x-\epsilon, x) \cup (x, x+\epsilon)\} \cap S = \emptyset$. The only way that x can be an element of ∂S in this circumstance is if $x \in S$. That is what we wished to prove.

For the other half of the theorem notice that if T is open then ${}^cT$ is closed. But then ${}^cT$ will contain all its boundary points, which are the same as the boundary points of T itself (why is this true?). Thus T can contain none of its boundary points. $\square$

Proposition: *Every nonisolated boundary point of a set S is an accumulation point of the set S.*

Proof: *This proof is treated in the exercises.* □

Definition

A subset S of the real numbers is called *bounded* if there is a positive number M such that $|s| \leq M$ for every element s of S. See the figure.


-M
M

The next result is one of the great theorems of nineteenth century analysis. It is essentially a restatement of the Bolzano–Weierstrass theorem of Section 2.2.

Theorem (Bolzano–Weierstrass)

Every bounded, infinite subset of $\mathbb{R}$ has an accumulation point.

Proof: Let S be a bounded, infinite set of real numbers. Let $\{a_j\}$ be a sequence of distinct elements of S. By an earlier theorem, there is a subsequence $\{a_{j_k}\}$ that converges to a limit α. Then α is an accumulation point of S. □

Corollary

Let $S \subseteq \mathbb{R}$ be a nonempty, closed, and bounded set. If $\{a_j\}$ is any sequence in S, then there is a Cauchy subsequence $\{a_{j_k}\}$ that converges to an element of S.

Proof: Merely combine the Bolzano–Weierstrass theorem with a proposition of the last section. □

Example: Consider the set $\{\sin j\}$. This set of real numbers is bounded by 1. By the Bolzano–Weierstrass theorem, it therefore has an accumulation point. So there is a sequence $\{\sin j_k\}$ that converges to some limit point p, even though it would be difficult to say precisely what that sequence is. □