Math 4111
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Figure: This is your instructor.

We are now in a position to give a complete description of all open sets.

Proposition: *Let $U \subseteq \mathbb{R}$ be a nonempty open set. Then there are either finitely many or countably many pairwise disjoint open intervals I_j such that*

$$U = \bigcup_{j=1}^{\infty} I_j \, .$$

See the next figure.



Proof: Assume that U is an open subset of the real line. We define an equivalence relation on the set U. The resulting equivalence classes will be the open intervals I_j.

Let a and b be elements of U. We say that a is related to b if all real numbers between a and b are also elements of U. It is obvious that this relation is both reflexive and symmetric. For transitivity notice that if a is related to b and b is related to c then (assuming that a, b, c are distinct) one of the numbers a, b, c must lie between the other two. Assume for simplicity that $a < b < c$. Then all numbers between a and c lie in U, for all such numbers are either between a and b or between b and c or are b itself. Thus a is related to c. (The other possible orderings of a, b, c are left for you to consider.)

Thus we have an equivalence relation on the set U. Call the equivalence classes $\{U_\alpha\}_{\alpha \in A}$. We claim that each U_α is an open interval. In fact if a, b are elements of some U_α then all points between a and b are in U. But then a moment's thought shows that each of those "in between" points is related to both a and b. Therefore all points between a and b are elements of U_α. We conclude that U_α is an interval. Is it an *open* interval?

Let $x \in U_\alpha$. Then $x \in U$ so that there is an open interval $I = (x - \epsilon, x + \epsilon)$ contained in U. But x is related to all the elements of I; it follows that $I \subseteq U_\alpha$. Therefore U_α is open.

We have exhibited the set U as a union of open intervals. These intervals are pairwise disjoint because they arise as the equivalence classes of an equivalence relation. Finally, each of these open intervals contains a (different) rational number (why?). Therefore there can be at most countably many of the intervals U_α. $\square$

It is worth noting, and we shall learn more about this fact in Chapter 12, that there is no structure theorem for open sets (like the one that we just proved) in dimension 2 and higher. The geometry of Euclidean space gets *much* more complicated as the dimension increases.

Definition

A subset $F \subseteq \mathbb{R}$ is called *closed* if the complement $\mathbb{R} \setminus F$ is open. See the next figure.



Example

The set $[0, 1]$ is closed. For its complement is

$$(-\infty, 0) \cup (1, \infty),$$

which is open.

Example

An interval of the form $[a, b] = \{x : a \leq x \leq b\}$ is closed. For its complement is $(-\infty, a) \cup (b, \infty)$, which is the union of two open intervals.

The finite set $A = \{-4, -2, 5, 13\}$ is closed because its complement is

$$(-\infty, -4) \cup (-4, -2) \cup (-2, 5) \cup (5, 13) \cup (13, \infty),$$

which is open.

The set $B = \{1, 1/2, 1/3, 1/4, \dots\} \cup \{0\}$ is closed, for its complement is the set

$$(-\infty, 0) \cup \left\{ \bigcup_{j=1}^{\infty} (1/(j+1), 1/j) \right\} \cup (1, \infty),$$

which is open.

Verify for yourself that if the point 0 is omitted from the set B, then the set is no longer closed. □

Roughly speaking, a closed set is a set that contains all its limit points. An open set is just the opposite—open sets tend not to contain their limit points. The discussion below will make these ideas more precise.

Remark: A common mistake that students make is to suppose that every set is either open or closed. This is not true. For instance, the set $[0, 1) = \{x \in \mathbb{R} : 0 \leq x < 1\}$ is neither open nor closed.

Proposition: *If E_α are closed sets, for α in some (possibly uncountable) index set A, then*

$$E = \bigcap_{\alpha \in A} E_\alpha$$

is closed.

Proof: This is just the contrapositive of an earlier proposition: if U_α is the complement of E_α, each α, then U_α is open. Then $U = \cup U_\alpha$ is also open. But then

$$E = \bigcap E_\alpha = \bigcap {}^c(U_\alpha) = {}^c\left(\bigcup U_\alpha\right) = {}^c U$$

is closed. Here ${}^c S$ denotes the complement of a set S. $\square$

The fact that the set B in the last example is closed, but that $B \setminus \{0\}$ is not, is placed in perspective by the next proposition.

Proposition: *Let S be a set of real numbers. Then S is closed if and only if every Cauchy sequence $\{s_j\}$ of elements of S has a limit point which is also an element of S.*

Proof: First suppose that S is closed and let $\{s_j\}$ be a Cauchy sequence in S. We know, since the reals are complete, that there is an element $s \in \mathbb{R}$ such that $s_j \to s$. The point of this half of the proof is to see that $s \in S$. If this statement were false then $s \in T = \mathbb{R} \setminus S$. But T must be open since it is the complement of a closed set. Thus there is an $\epsilon > 0$ such that the interval $I = (s - \epsilon, s + \epsilon) \subseteq T$. This means that no element of S lies in I. In particular, $|s - s_j| \geq \epsilon$ for every j. This contradicts the statement that $s_j \to s$. We conclude that $s \in S$.

Conversely, assume that every Cauchy sequence in S has its limit in S. If S were not closed then its complement would not be open. Hence there would be a point $t \in \mathbb{R} \setminus S$ with the property that no interval $(t - \epsilon, t + \epsilon)$ lies in $\mathbb{R} \setminus S$. In other words, $(t - \epsilon, t + \epsilon) \cap S \neq \emptyset$ for every $\epsilon > 0$. Thus for $j = 1, 2, 3, \ldots$ we may choose a point $s_j \in (t - 1/j, t + 1/j) \cap S$. It follows that $\{s_j\}$ is a sequence of elements of S that converge to $t \in \mathbb{R} \setminus S$. That contradicts our hypothesis. We conclude that S must be closed. $\square$

Definition

Let S be a subset of $\mathbb{R}$. A point x is called an *accumulation point* of S if every neighborhood of x contains infinitely many distinct elements of S. See the next figure. In particular, x is an accumulation point of S if it is the limit of a sequence of distinct elements in S.

The last proposition tells us that closed sets are characterized by the property that they contain all of their accumulation points.

