Math 4111
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Figure: This is your instructor.

To specify a topology on a set is to describe certain subsets that will play the role of neighborhoods. These sets are called *open sets*. Our purpose here is to be able to study the "shape" of a set without worrying about its rigid properties. People like to joke that a mathematician does not know a coffee cup from a donut because they both have the same shape—a loop with a hole in the middle. See the first figure. The purpose of this chapter is to make these ideas precise.

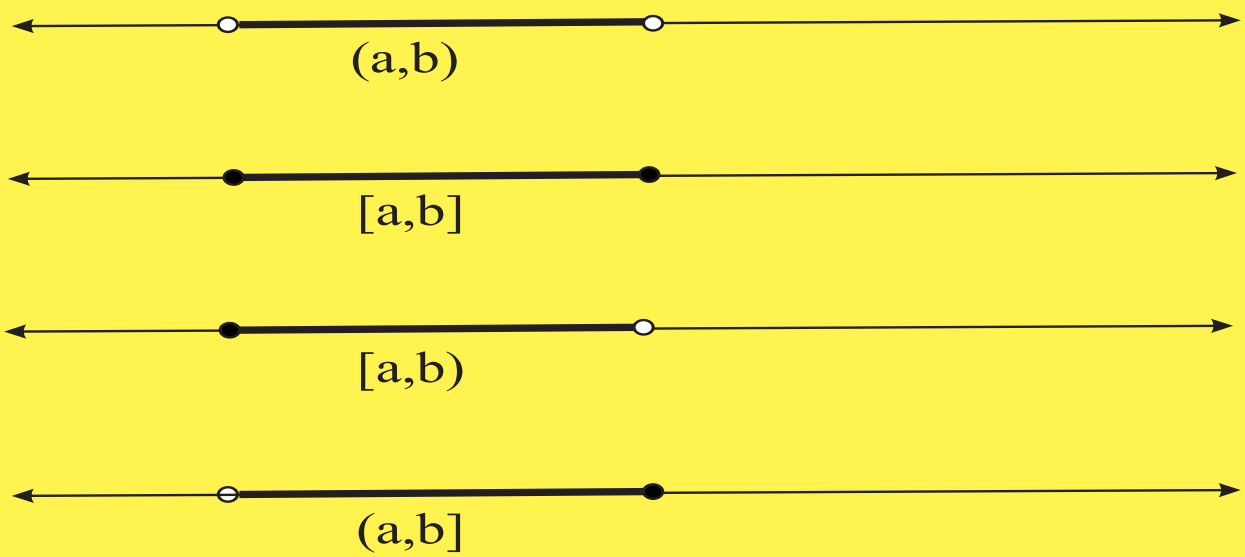


Figure: A coffee cup and a donut.

In what follows, we will use "interval notation": If $a \leq b$ are real numbers then we define

$$\begin{aligned}
(a,b) &= \{x \in \mathbb{R} : a < x < b\}, \\
[a,b] &= \{x \in \mathbb{R} : a \leq x \leq b\}, \\
[a,b) &= \{x \in \mathbb{R} : a \leq x < b\}, \\
(a,b] &= \{x \in \mathbb{R} : a < x \leq b\}.
\end{aligned}$$

Intervals of the form (a,b) are called *open*. Those of the form $[a,b]$ are called *closed*. The other two are termed *half-open* or *half-closed*. See the next figure.


(a,b)
[a,b]
[a,b)
(a,b]

Now we extend the terms "open" and "closed" to more general sets.

Definition

A set $U \subseteq \mathbb{R}$ is called *open* if, for each $x \in U$, there is an $\epsilon > 0$ such that the interval $(x - \epsilon, x + \epsilon)$ is contained in U. See the next figure.

Remark: The interval $(x - \epsilon, x + \epsilon)$ is frequently termed a *neighborhood* of x.

Example

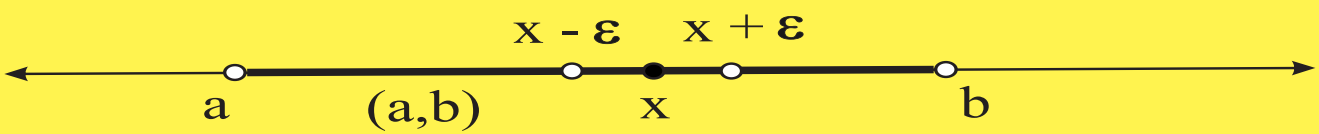
The set $U = \{x \in \mathbb{R} : |x-3| < 2\}$ is open. To see this, choose a point $x \in U$. Let $\epsilon = 2 - |x-3| > 0$. [Notice that we are choosing ϵ to be the distance of x to the boundary of U.] Then we claim that the interval $I = (x-\epsilon, x+\epsilon) \subseteq U$.

For, if $t \in I$, then

$$\begin{aligned} |t-3| &\leq |t-x| + |x-3| \\ &< \epsilon + |x-3| \\ &= (2 - |x-3|) + |x-3| = 2\,. \end{aligned}$$

But this means that $t \in U$.

We have shown that $t \in I$ implies $t \in U$. Therefore $I \subseteq U$. It follows from the definition that U is open. □


x - ε
x + ε
a
(a,b)
x
b

Remark: The way to think about the definition of open set is that a set is open when none of its elements is at the "edge" of the set—each element is surrounded by other elements of the set, indeed a whole interval of them. Contrast the last figure with the next figure. The remainder of this section will make these comments precise.


point at edge of set

Proposition: *If U_α are open sets, for α in some (possibly uncountable) index set A, then*

$$U = \bigcup_{\alpha \in A} U_\alpha$$

is open.

Proof: Let $x \in U$. By definition of union, the point x must lie in some U_α. But U_α is open. Therefore there is an interval $I = (x - \epsilon, x + \epsilon)$ such that $I \subseteq U_\alpha$. Therefore $I \subseteq U$. This proves that U is open. $\square$

Proposition: *If $U_1, U_2, \ldots, U_k$ are open sets then the set*

$$V = \bigcap_{j=1}^{k} U_j$$

is also open.

Proof: Let $x \in V$. Then $x \in U_j$ for each j. Since each U_j is open there is for each j a positive number ϵ_j such that $I_j = (x - \epsilon_j, x + \epsilon_j)$ lies in U_j. Set $\epsilon = \min\{\epsilon_1, \ldots, \epsilon_k\}$. Then $\epsilon > 0$ and $(x - \epsilon, x + \epsilon) \subseteq U_j$ for every j. But that just means that $(x - \epsilon, x + \epsilon) \subseteq V$. Therefore V is open. $\square$

Notice the difference between these two propositions: arbitrary unions of open sets are open. But, in order to guarantee that an intersection of open sets is still open, we had to assume that we were only intersecting finitely many such sets. If there were infinitely many sets then the minimum of the ϵ_j could be 0.

To understand this matter better, bear in mind the example of the open sets

$$U_j = \left(-\frac{1}{j}, \frac{1}{j}\right) \ , \quad j = 1, 2, \ldots \ .$$

Each of the sets U_j is open, but the intersection of the sets U_j is the singleton $\{0\}$, which is *not* open.

The same analysis as in the first example shows that, if $a < b$, then the interval (a, b) is an open set. On the other hand, intervals of the form $(a, b]$ or $[a, b)$ or $[a, b]$ are *not* open. In the first instance, the point b is the center of no interval $(b - \epsilon, b + \epsilon)$ contained in $(a, b]$. Think about the other two intervals to understand why they are not open. We call intervals of the form (a, b) *open intervals*.