Math 4111
October 16, 2020 Lecture

Steven G. Krantz



Figure: This is your instructor.

Operations on Series

Some operations on series, such as addition, subtraction, and scalar multiplication, are straightforward. Others, such as multiplication, entail subtleties. This section treats all these matters.

Proposition: *Let*

$$\sum_{j=1}^{\infty} a_j \quad \text{and} \quad \sum_{j=1}^{\infty} b_j$$

be convergent series of real or complex numbers; assume that the series sum to limits α and β respectively. Then

(a) *The series $\sum_{j=1}^{\infty}(a_j + b_j)$ converges to the limit $\alpha + \beta$.*

(b) *If c is a constant then the series $\sum_{j=1}^{\infty} c \cdot a_j$ converges to $c \cdot \alpha$.*

Proof: We shall prove assertion **(a)** and leave the easier assertion **(b)** as an exercise.

Pick $\epsilon > 0$. Choose an integer N_1 so large that $n > N_1$ implies that the partial sum $S_n \equiv \sum_{j=1}^{n} a_j$ satisfies $|S_n - \alpha| < \epsilon/2$. Choose N_2 so large that $n > N_2$ implies that the partial sum $T_n \equiv \sum_{j=1}^{n} b_j$ satisfies $|T_n - \beta| < \epsilon/2$. If U_n is the nth partial sum of the series $\sum_{j=1}^{\infty}(a_j + b_j)$ and if $n > N_0 \equiv \max(N_1, N_2)$ then

$$|U_n - (\alpha + \beta)| \leq |S_n - \alpha| + |T_n - \beta| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon\,.$$

Thus the sequence $\{U_n\}$ converges to $\alpha + \beta$. This proves part **(a)**. The proof of **(b)** is similar. □

In order to keep our discussion of multiplication of series as straightforward as possible, we deal at first with absolutely convergent series. Here a series $\sum_{j=1}^{\infty} a_j$ is *absolutely convergent* if $\sum_{j=1}^{\infty} |a_j| < \infty$. It is a basic fact, which follows from the Cauchy condition, that if $\sum_j a_j$ is absolutely convergent, then it is convergent.

It is convenient in this discussion to begin our sums at $j = 0$ instead of $j = 1$. If we wish to multiply

$$\sum_{j=0}^{\infty} a_j \quad \text{and} \quad \sum_{j=0}^{\infty} b_j \,,$$

then we need to specify what the partial sums of the product series should be. An obvious necessary condition that we wish to impose is that, if the first series converges to α and the second converges to β, then the product series, whatever we define it to be, should converge to $\alpha \cdot \beta$.

The naive method for defining the summands of the product series $\sum_j c_j$ is to let $c_j = a_j \cdot b_j$. However, a glance at the product of two partial sums of the given series shows that such a definition would be ignoring the distributivity of multiplication over addition.

Cauchy's idea was that the summands for the product series should be

$$c_m \equiv \sum_{j=0}^{m} a_j \cdot b_{m-j} \, .$$

This particular form for the summands can be easily motivated using power series considerations (which we shall provide later in the course). For now we concentrate on verifying that this "Cauchy product" of two series really works.

Theorem

Let $\sum_{j=0}^{\infty} a_j$ and $\sum_{j=0}^{\infty} b_j$ be two absolutely convergent series which converge to limits α and β respectively. Define the series $\sum_{m=0}^{\infty} c_m$ with summands $c_m = \sum_{j=0}^{m} a_j \cdot b_{m-j}$. Then the series $\sum_{m=0}^{\infty} c_m$ converges absolutely to $\alpha \cdot \beta$.

Proof: Let $A_n, B_n,$ and C_n be the partial sums of the three series in question. We calculate that

$$\begin{aligned} C_n &= (a_0 b_0) + (a_0 b_1 + a_1 b_0) + (a_0 b_2 + a_1 b_1 + a_2 b_0) \\ &\qquad + \cdots + (a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0) \\ &= a_0 \cdot B_n + a_1 \cdot B_{n-1} + a_2 \cdot B_{n-2} + \cdots + a_n \cdot B_0 \,. \end{aligned}$$

We set $\lambda_n = B_n - \beta$, each n, and rewrite the last line as

$$\begin{aligned} C_n &= a_0(\beta + \lambda_n) + a_1(\beta + \lambda_{n-1}) + \cdots a_n(\beta + \lambda_0) \\ &= A_n \cdot \beta + [a_0 \lambda_n + a_1 \cdot \lambda_{n-1} + \cdots + a_n \cdot \lambda_0] \,. \end{aligned}$$

Denote the expression in square brackets by the symbol ρ_n. Suppose that we could show that $\lim_{n\to\infty} \rho_n = 0$. Then we would have

$$\begin{aligned}
\lim_{n\to\infty} C_n &= \lim_{n\to\infty} (A_n \cdot \beta + \rho_n) \\
&= (\lim_{n\to\infty} A_n) \cdot \beta + (\lim_{n\to\infty} \rho_n) \\
&= \alpha \cdot \beta + 0 \\
&= \alpha \cdot \beta \,.
\end{aligned}$$

Thus it is enough to examine the limit of the expressions ρ_n.

Since $\sum_{j=1}^{\infty} a_j$ is absolutely convergent, we know that $A = \sum_{j=1}^{\infty} |a_j|$ is a finite number. Choose $\epsilon > 0$. Since $\sum_{j=1}^{\infty} b_j$ converges to β it follows that $\lambda_n \to 0$. Thus we may choose an integer $N > 0$ such that $n > N$ implies that $|\lambda_n| < \epsilon$. Thus, for $n = N + k, k > 0$, we may estimate

$$\begin{aligned}
|\rho_{N+k}| &\leq |\lambda_0 a_{N+k} + \lambda_1 a_{N+k-1} + + \cdots + \lambda_N a_k| \\
&\qquad + |\lambda_{N+1} a_{k-1} + \lambda_{N+2} a_{k-2} + \cdots + \lambda_{N+k} a_0| \\
&\leq |\lambda_0 a_{N+k} + \lambda_1 a_{N+k-1} + + \cdots + \lambda_N a_k| \\
&\qquad + \max_{p \geq 1}\{|\lambda_{N+p}|\} \cdot (|a_{k-1}| + |a_{k-2}| + \cdots + |a_0|) \\
&\leq (N+1) \cdot \max_{\ell \geq k} |a_\ell| \cdot \max_{0 \leq j \leq N} |\lambda_j| + \epsilon \cdot A \,.
\end{aligned}$$

In this last estimate, we have used the fact (for the first term in absolute values) that **(a)** there are $N+1$ summands, **(b)** the a terms all have index at least k, and **(c)** the λ terms have index between 0 and N. The second term (the "max" term) is easy to estimate because of our bound on λ_n.

With N fixed, we let $k \to \infty$ in the last inequality. Since $\max_{\ell \geq k} |a_\ell| \to 0$, we find that

$$\limsup_{n \to \infty} |\rho_n| \leq \epsilon \cdot A \,.$$

Since $\epsilon > 0$ was arbitrary, we conclude that

$$\lim_{n \to \infty} |\rho_n| \to 0 \,.$$

This completes the proof. $\square$

Notice that, in the proof of the theorem, we really only used the fact that one of the given series was absolutely convergent, not that both were absolutely convergent. Some hypothesis of this nature is necessary, as the following example shows.

Example

Consider the Cauchy product of the two conditionally convergent series

$$\sum_{j=0}^{\infty} \frac{(-1)^j}{\sqrt{j+1}} \quad \text{and} \quad \sum_{j=0}^{\infty} \frac{(-1)^j}{\sqrt{j+1}}.$$

Observe that

$$\begin{aligned} c_m &= \frac{(-1)^0(-1)^m}{\sqrt{1}\sqrt{m+1}} + \frac{(-1)^1(-1)^{m-1}}{\sqrt{2}\sqrt{m}} + \cdots \\ &\qquad + \frac{(-1)^m(-1)^0}{\sqrt{m+1}\sqrt{1}} \\ &= \sum_{j=0}^{m} (-1)^m \frac{1}{\sqrt{(j+1)\cdot(m+1-j)}}\,. \end{aligned}$$

However, for $0 \le j \le m$,

$$(j+1)\cdot(m+1-j) \le (m+1)\cdot(m+1) = (m+1)^2\,.$$

Thus

$$|c_m| \ge \sum_{j=0}^{m} \frac{1}{m+1} = 1\,.$$

We thus see that the terms of the series $\sum_{m=0}^{\infty} c_m$ do not tend to zero, so the series cannot converge.