Math 4111
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Figure: This is your instructor.

3.4 Some Special Series

Exercises: 1–5.

We begin with a series that defines a special constant of mathematical analysis.

Definition

The series

$$\sum_{j=0}^{\infty} \frac{1}{j!},$$

where $j! \equiv j \cdot (j-1) \cdot (j-2) \cdots 1$ for $j \geq 1$ and $0! \equiv 1$, is convergent (by the Ratio Test, for instance). Its sum is denoted by the symbol e in honor of the Swiss mathematician Léonard Euler, who first studied it (see also Example 2.47, where the number e is studied by way of a sequence). We shall see in the next Proposition that these two approaches to the number e are equivalent.

Like the number π, to be considered later in this book, the number e is one which arises repeatedly in a number of contexts in mathematics. It has many special properties. We first relate the series definition of e to the sequence definition:

Proposition

The limit

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

exists and equals e.

Proof: We need to compare the quantities

$$A_N \equiv \sum_{j=0}^{N} \frac{1}{j!} \quad \text{and} \quad B_N \equiv \left(1 + \frac{1}{N}\right)^N .$$

We use the binomial theorem to expand B_N :

$$\begin{aligned} B_N &= 1 + \frac{N}{1} \cdot \frac{1}{N} + \frac{N \cdot (N-1)}{2 \cdot 1} \cdot \frac{1}{N^2} + \frac{N \cdot (N-1) \cdot (N-2)}{3 \cdot 2 \cdot 1} \cdot \frac{1}{N^3} \\ &\quad + \cdots + \frac{N}{1} \cdot \frac{1}{N^{N-1}} + 1 \cdot \frac{1}{N^N} \\ &= 1 + 1 + \frac{1}{2!} \cdot \frac{N-1}{N} + \frac{1}{3!} \cdot \frac{N-1}{N} \cdot \frac{N-2}{N} + \cdots \\ &\quad + \frac{1}{(N-1)!} \cdot \frac{N-1}{N} \cdot \frac{N-2}{N} \cdots \frac{2}{N} \\ &\quad + \frac{1}{N!} \cdot \frac{N-1}{N} \cdot \frac{N-2}{N} \cdots \frac{1}{N} \end{aligned}$$

$$= 1+1+\frac{1}{2!}\cdot\left(1-\frac{1}{N}\right)+\frac{1}{3!}\cdot\left(1-\frac{1}{N}\right)\cdot\left(1-\frac{2}{N}\right)+\cdots$$
$$+\frac{1}{(N-1)!}\cdot\left(1-\frac{1}{N}\right)\cdot\left(1-\frac{2}{N}\right)\cdots\left(1-\frac{N-2}{N}\right)$$
$$+\frac{1}{N!}\cdot\left(1-\frac{1}{N}\right)\cdot\left(1-\frac{2}{N}\right)\cdots\left(1-\frac{N-1}{N}\right).$$

Notice that every summand that appears in this last equation is positive. Thus, for $0 \le M \le N$,

$$B_N \ge 1+1+\frac{1}{2!}\cdot\left(1-\frac{1}{N}\right)+\frac{1}{3!}\cdot\left(1-\frac{1}{N}\right)\cdot\left(1-\frac{2}{N}\right)$$
$$+\cdots+\frac{1}{M!}\cdot\left(1-\frac{1}{N}\right)\left(1-\frac{2}{N}\right)\cdots\left(1-\frac{M-1}{N}\right).$$

In this last inequality we hold M fixed and let N tend to infinity. The result is that

$$\liminf_{N\to\infty} B_N \ge 1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{M!} = A_M\,.$$

Now, as $M \to \infty$, the quantity A_M converges to e (by the *definition* of e). So we obtain

$$\liminf_{N\to\infty} B_N \geq e\,. \tag{1}$$

On the other hand, our expansion for B_N allows us to observe that $B_N \leq A_N$. Thus

$$\limsup_{N\to\infty} B_N \leq e\,. \tag{2}$$

Combining (1) and (2) we find that

$$e \leq \liminf_{N\to\infty} B_N \leq \limsup_{N\to\infty} B_N \leq e$$

hence that $\lim_{N\to\infty} B_N$ exists and equals e. This is the desired result. □

Remark

The last proof illustrates the value of the concepts of lim inf and lim sup. For we do not know in advance that the limit of the expressions B_N exists, much less that the limit equals e. However, the lim inf and the lim sup always exist. So we estimate those instead, and find that they are equal and that they equal e.

The next result tells us how rapidly the partial sums A_N of the series defining e converge to e. This is of theoretical interest, but will also be applied to determine the irrationality of e.

Proposition

With A_N as above, we have that

$$0 < e - A_N < \frac{1}{N \cdot N!} .$$

Proof: Observe that

$$\begin{aligned} e - A_N &= \frac{1}{(N+1)!} + \frac{1}{(N+2)!} + \frac{1}{(N+3)!} + \cdots \\ &= \frac{1}{(N+1)!} \cdot \left(1 + \frac{1}{N+2} + \frac{1}{(N+2)(N+3)} + \cdots\right) \\ &< \frac{1}{(N+1)!} \cdot \left(1 + \frac{1}{N+1} + \frac{1}{(N+1)^2} + \cdots\right). \end{aligned}$$

Now the expression in parentheses is a geometric series. It sums to $(N+1)/N$. Since $A_N < e$, we have

$$e - A_N = |e - A_N|$$

hence

$$|e - A_N| < \frac{1}{N \cdot N!},$$

proving the result. □

Next we prove that e is an irrational number.

Theorem

Euler's number e is irrational.

Proof: Suppose to the contrary that e is rational. Then $e = p/q$ for some positive integers p and q. By the preceding proposition,

$$0 < e - A_q < \frac{1}{q \cdot q!}$$

or

$$0 < q! \cdot (e - A_q) < \frac{1}{q}\,. \tag{3}$$

Now

$$e - A_q = \frac{p}{q} - \left(1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{q!}\right).$$

Hence,

$$q! \cdot (e - A_q)$$

is an integer. But then equation (3) says that this integer lies between 0 and $1/q$.

In particular, this integer lies strictly between 0 and 1. That, of course, is impossible. So e must be irrational. □

It is a general principle of number theory that a real number that can be approximated *too rapidly* by rational numbers (the degree of rapidity being measured in terms of powers of the denominators of the rational numbers) must be irrational.

Under suitable conditions an even stronger conclusion holds: namely, the number in question turns out to be *transcendental*. A transcendental number is one which is not the solution of any polynomial equation with integer coefficients.

The subject of transcendental numbers is explored in the exercises. The exercises also contain a sketch of a proof that e is transcendental.

In Exercise **5** of Section 2.4, we briefly discussed Euler's number γ. Both this special number and also the more commonly encountered number π arise in many contexts in mathematics. It is unknown whether γ is rational or irrational.

The number π is known to be transcendental, but it is unknown whether $\pi + e$ is transcendental.

In recent years, questions about the the irrationality and transcendence of various numbers have become a matter of practical interest. For these properties prove to be useful in making and breaking secret codes, and in encrypting information so that it is accessible to some users but not to others.

Recall that

$$S_N \equiv \sum_{j=1}^{N} j = \frac{N \cdot (N+1)}{2}.$$

We conclude this section with a method for summing higher powers of j.

Suppose that we wish to calculate

$$S_{k,N} \equiv \sum_{j=1}^{N} j^k$$

for some positive integer k exceeding 1. We may proceed as follows: Write

$$(j+1)^{k+1} - j^{k+1} = \left[j^{k+1} + (k+1) \cdot j^k + \frac{(k+1) \cdot k}{2} \cdot j^{k-1} + \cdots + \frac{(k+1) \cdot k}{2} \cdot j^2 + (k+1) \cdot j + 1 \right] - j^{k+1}$$

$$= (k+1)\cdot j^k + \frac{(k+1)\cdot k}{2}\cdot j^{k-1} + \cdots + \frac{(k+1)\cdot k}{2}\cdot j^2 + (k+1)\cdot j + 1\,.$$

Summing from $j = 1$ to $j = N$ yields

$$\sum_{j=1}^{N}\left\{(j+1)^{k+1} - j^{k+1}\right\} = (k+1)\cdot S_{k,N} + \frac{(k+1)\cdot k}{2}\cdot S_{k-1,N} + \cdots + \frac{(k+1)\cdot k}{2}\cdot S_{2,N} + (k+1)\cdot S_{1,N} + N\,.$$

The sum on the left collapses to $(N+1)^{k+1} - 1$. We may solve for $S_{k,N}$ and obtain

$$S_{k,N} = \frac{1}{k+1} \cdot \left[(N+1)^{k+1} - 1 - N - \frac{(k+1) \cdot k}{2} \cdot S_{k-1,N} \right.$$
$$\left. - \cdots - \frac{(k+1) \cdot k}{2} \cdot S_{2,N} - (k+1) \cdot S_{1,N} \right] .$$

We have succeeded in expressing $S_{k,N}$ in terms of $S_{1,N}, S_{2,N}, \ldots, S_{k-1,N}$. Thus we may inductively obtain formulas for $S_{k,N}$, any k. It turns out that

$$S_{1,N} = \frac{N(N+1)}{2}$$
$$S_{2,N} = \frac{N(N+1)(2N+1)}{6}$$
$$S_{3,N} = \frac{N^2(N+1)^2}{4}$$
$$S_{4,N} = \frac{(N+1)N(2N+1)(3N^2+3N-1)}{30}$$

These formulas are treated in further detail in the exercises.