Math 4111
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Figure: This is your instructor.

Example

Show that the the series $\displaystyle\sum_{j=1}^{\infty} \frac{\sin j}{j}$. converges.

Proof: We take $a_j = \sin j$ and $b_j = 1/j$. Observe that

$$\cos(j + 1/2) = \cos j \cdot \cos 1/2 - \sin j \cdot \sin 1/2, \quad \text{and}$$

$$\cos(j - 1/2) = \cos j \cdot \cos 1/2 + \sin j \cdot \sin 1/2\,.$$

Subtracting these equations and solving for $\sin j$ yields that

$$\sin j = \frac{\cos(j - 1/2) - \cos(j + 1/2)}{2 \cdot \sin 1/2}.$$

We conclude that

$$A_N = \sum_{j=1}^{N} a_j = \sum_{j=1}^{N} \frac{\cos(j - 1/2) - \cos(j + 1/2)}{2 \cdot \sin 1/2}.$$

Of course this sum collapses and we see that

$$A_N = \frac{-\cos(N + 1/2) + \cos 1/2}{2 \cdot \sin 1/2}.$$

Thus $|A_N| \leq \dfrac{2}{2 \cdot \sin 1/2} = \dfrac{1}{\sin 1/2}$, independent of N.
Thus the hypotheses of Abel's test are verified and the series $\displaystyle\sum_{j=1}^{\infty} \frac{\sin j}{j}$ converges. □

Exercise: Show that both the series

$$\sum_{j=1}^{\infty} \frac{|\sin j|}{j} \quad \text{and} \quad \sum_{j=1}^{\infty} \frac{\sin^2 j}{j}$$

diverge. Hint: Combine two terms with indices $2k-1$ and $2k$.

We turn next to the topic of absolute and conditional convergence.

Definition

A series of real or complex numbers $\sum_{j=1}^{\infty} a_j$ is said to be

absolutely convergent if $\sum_{j=1}^{\infty} |a_j|$ converges.

For example, the series $\sum_{j=1}^{\infty} \frac{\cos j}{j^2}$ is absolute convergent.

The following fact is an immediate consequence of the Comparison Test.

Proposition

If the series $\sum_{j=1}^{\infty} a_j$ is absolutely convergent, then it is convergent.

Definition

A series $\sum_{j=1}^{\infty} a_j$ is said to be *conditionally convergent* if $\sum_{j=1}^{\infty} a_j$ converges, but it does not converge absolutely.

We see that absolutely convergent series are convergent but the next example shows that the converse is not true.

Example

The series

$$\sum_{j=1}^{\infty} \frac{(-1)^j}{j}$$

converges by the Alternating Series Test. However, it is not absolutely convergent because the harmonic series $\sum_{j=1}^{\infty} \frac{1}{j}$ diverges. □

There is a remarkable robustness result for absolutely convergent series that fails dramatically for conditionally convergent series. This result is enunciated in the next theorem. We first need a definition.

Definition

Let $\sum_{j=1}^{\infty} a_j$ be a given series. Let $\{p_j\}_{j=1}^{\infty}$ be a sequence in which every positive integer occurs once and only once (but not necessarily in the usual order). We call $\{p_j\}$ a *permutation* of the natural numbers.

Then the series $\displaystyle\sum_{j=1}^{\infty} a_{p_j}$ is said to be a *rearrangement* of the given series.

Theorem (Riemann, Weierstrass)

Let $\sum_{j=1}^{\infty} a_j$ be an absolutely convergent series and let ℓ be its sum. Then every rearrangement of the series $\sum_{j=1}^{\infty} a_j$ converges also to ℓ. If the real series $\sum_{j=1}^{\infty} b_j$ is conditionally convergent and β is any real number or $\pm\infty$, then there is a rearrangement of the series that converges to β.

Proof: We prove the first assertion here and explore the second in the exercises.

Let us choose a rearrangement of the given series and denote it by $\sum_{j=1}^{\infty} a_{p_j}$, where p_j is a permutation of the positive integers. Pick $\epsilon > 0$. By the hypothesis that the original series converges absolutely we may choose an integer $N > 0$ such that $N < m \leq n < \infty$ implies that

$$\sum_{j=m}^{n} |a_j| < \epsilon \,. \tag{2}$$

[The presence of the absolute values in the left side of this inequality will prove crucial in a moment.] Choose a positive integer M such that $M \geq N$ and the integers $1, \ldots, N$ are all contained in the list $p_1, p_2, \ldots, p_M$. If $K > M$ then the partial sum $\sum_{j=1}^{K} a_j$ will trivially contain the summands $a_1, a_2, \ldots a_N$. Also the partial sum $\sum_{j=1}^{K} a_{p_j}$ will contain the summands $a_1, a_2, \ldots a_N$.

It follows that

$$\sum_{j=1}^{K} a_j - \sum_{j=1}^{K} a_{p_j}$$

will contain only summands *after* the Nth one in the original series. By inequality (2) we may conclude that

$$\left| \sum_{j=1}^{K} a_j - \sum_{j=1}^{K} a_{p_j} \right| \leq \sum_{j=N+1}^{\infty} |a_j| \leq \epsilon \, .$$

We conclude that the rearranged series converges; and it converges to the same sum as the original series. □

Exercise: Given that the sequence (a_n) converges to α, and (p_n) is a permutation of $\mathbb{N}$. Show that (a_{p_n}) also converges to α.

Exercise: Suppose that $\sum_{n=1}^{\infty} b_n$ is a divergent series with positive terms and let s_n be its nth partial sum.

(a). Show that $\sum_{n=1}^{\infty} b_n/s_n$ diverges.

(b). Show that there is a divergent series $\sum_{n=1}^{\infty} c_n$ with positive terms such that $\lim_{n\to\infty} c_n/b_n = 0$.

(c). Show that $\sum_{n=1}^{\infty} b_n/s_n^2$ converges.