Math 4111
October 9, 2020 Lecture

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Figure: This is your instructor.

Now we have the first nontrivial convergence test for series.

Theorem (The Root Test)

Consider the series $\sum_{j=1}^{\infty} a_j$ *and let* $r = \limsup_{j\to\infty} |a_j|^{1/j}$.

(a) If $r < 1$, *then the series* $\sum_{j=1}^{\infty} a_j$ *converges.*

(b) If $r > 1$, *then the series* $\sum_{j=1}^{\infty} a_j$ *diverges.*

Proof: (a). By our hypothesis, for $\beta = \frac{r+1}{2} < 1$ there is an integer $N > 1$ such that, for all $j > N$, it holds that $|a_j|^{1/j} < \beta$, namely, $|a_j| < \beta^j$. Since $0 < \beta < 1$, by the Geometric Series Test, $\sum_{j=1}^{\infty} \beta^j$ converges. By the Comparison Test, $\sum_{j=1}^{\infty} a_j$ converges.

(b). When $r > 1$, it can be seen that there are infinitely many positive integers j such that $|a_j|^{1/j} > \frac{1+r}{2} = \beta > 1$, that is, $|a_j| > \beta^j$. Thus (a_j) is unbounded, and $\sum_{j=1}^{\infty} a_j$ diverges. □

Note that the Root Test is inconclusive if $r = 1$. Indeed, for both $\sum_{n=1}^{\infty} 1/n$ and $\sum_{n=1}^{\infty} 1/n^2$, we have $r = 1$, but only the second series converges.

Example

Determine if the series $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ converges.

Solution. Let $a_n = \dfrac{n^2}{2^n}$. Then

$$|a_n|^{1/n} = \frac{\left(n^{1/n}\right)^2}{2}.$$

As $n \to \infty$, we see that

$$\limsup_{n\to\infty} |a_n|^{1/n} = \frac{1}{2} < 1.$$

By the Root Test, the series converges. □

Note that the Root Test is not easy to apply when complicated factors such as $n!$ are involved, as in $\sum_{n=1}^{\infty} \dfrac{5^n}{n!}$.

Theorem (The Ratio Test)

Consider a series $\sum_{j=1}^{\infty} a_j$. *If* $\limsup_{j\to\infty} \left|\frac{a_{j+1}}{a_j}\right| < 1$, *then the series* $\sum_{j=1}^{\infty} a_j$ *converges.*

Proof: Let $\lambda = \limsup_{j\to\infty} \left|\frac{a_{j+1}}{a_j}\right| < 1$. Select a real number μ such that $\lambda < \mu < 1$. By the definition of lim sup, there is an N so large that if $j > N$ then $\left|\frac{a_{j+1}}{a_j}\right| < \mu$. This may be rewritten as $|a_{j+1}| < \mu \cdot |a_j|$, for all $j \geq N$. Thus (much as in the proof of the Root Test) we have for $k \geq 0$ that
$|a_{N+k}| \leq \mu \cdot |a_{N+k-1}| \leq \mu \cdot \mu \cdot |a_{N+k-2}| \leq \cdots \leq \mu^k \cdot |a_N|$.
It is convenient to denote $N + k$ by $n, n \geq N$. Thus the last inequality reads

$$|a_n| < \mu^{n-N} \cdot |a_N|$$

or

$$|a_n|^{1/n} < \mu^{(n-N)/n} \cdot |a_N|^{1/n} .$$

Remembering that N has been fixed once and for all, we pass to the lim sup as $n \to \infty$. The result is

$$\limsup_{n\to\infty} |a_n|^{1/n} \leq \mu .$$

Since $\mu < 1$, we find that our series satisfies the hypotheses of the Root Test. Hence it converges. □

Remark. The proof of the Ratio Test shows that if a series passes the Ratio Test then it passes the Root Test (the converse is not true, as you will learn in Exercise **2**).

Why do we therefore learn the Ratio Test? The answer is that there are circumstances when the Ratio Test is easier to apply than the Root Test.

Note that $\limsup_{j\to\infty} \left|\frac{a_{j+1}}{a_j}\right| > 1$ does not imply that $\sum_{j=1}^{\infty} a_j$ diverges.

For instance, consider $\sum_{j=1}^{\infty} \frac{1}{(3+(-1)^j)^j}$.

However, we have

Theorem (The Ratio Test for Divergence)

If $\liminf_{j\to\infty} \left|\frac{a_{j+1}}{a_j}\right| > 1$, *then the series* $\sum_{j=1}^{\infty} a_j$ *diverges.*

For example, the series $\sum_{j=1}^{\infty} \frac{j!}{5^j}$ diverges as

$\lim_{j\to\infty} \left|\frac{a_{j+1}}{a_j}\right| = \infty > 1.$

Example

Show that the series $\sum_{j=1}^{\infty} \frac{2^j}{j!}$ converges.

Proof: We have $a_j = 2^j/j!$ and

$$\left|\frac{a_{j+1}}{a_j}\right| = \frac{2^{j+1}/(j+1)!}{2^j/j!} = \frac{2}{j+1}.$$

The lim sup of the last expression is 0. By the Ratio Test, the series converges. □

Example

Determine if $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ converges.

Solution. We have $a_n = \frac{n!}{n^n}$ and

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)!/(n+1)^{n+1}}{n!/n^n} = \frac{n^n}{(n+1)^n} = \frac{1}{(1+1/n)^n} \to \frac{1}{e} < 1.$$

By the Ratio Test, the series converges. □

The Integral Test

We conclude this section with the Integral Test.

Proposition (The Integral Test)

Let f be a continuous and decreasing function on $[0, \infty)$. The series $\sum_{j=1}^{\infty} f(j)$ converges if and only if the integral $\int_1^{\infty} f(x)\,dx$ converges.

Proof Outline. The improper integral $\int_1^{\infty} f(x)\,dx$ is the area below the curve $y = f(x)$ over the interval $[1, \infty]$. Observe that $\int_1^{\infty} f(x)\,dx \geq \sum_{j=2}^{\infty} f(j)$ and $\sum_{j=1}^{\infty} f(j) \geq \int_1^{\infty} f(x)\,dx$. □

Example

Let $p > 1$. Show that the series $\displaystyle\sum_{j=1}^{\infty} \frac{2}{j(\ln j)^p}$ converges.

Proof. Note that $\frac{1}{x(\ln x)^p}$ is continuous and decreasing on $[2, \infty)$, and

$$\int_2^{\infty} \frac{1}{x(\ln x)^p}\, dx = \lim_{N \to +\infty} \left. \frac{(\ln x)^{1-p}}{1-p} \right|_2^N = \frac{(\ln 2)^{1-p}}{p-1} \quad \text{converges.}$$

Therefore, the series converges. □

In this section we consider convergence tests for series which depend on cancellation among the terms of the series. One of the most profound of these depends on a technique called *summation by parts*. You may wonder whether this process is at all related to the "integration by parts" procedure that you learned in calculus—it has a similar form. Indeed it will turn out (and we shall see the details of this assertion as the book develops) that summing a series and performing an integration are two aspects of the same limiting process. The summation by parts method is merely our first glimpse of this relationship.

Proposition (Summation by Parts)

Let $\{a_j\}_{j=0}^{\infty}$ and $\{b_j\}_{j=0}^{\infty}$ be two sequences of real or complex numbers. For $N = 0, 1, 2, \ldots$ set

$$A_N = \sum_{j=0}^{N} a_j$$

(we adopt the convention that $A_{-1} = 0$). Then, for any $0 \leq m \leq n < \infty$, it holds that

$$\sum_{j=m}^{n} a_j \cdot b_j = [A_n \cdot b_n - A_{m-1} \cdot b_m] + \sum_{j=m}^{n-1} A_j \cdot (b_j - b_{j+1}) .$$

Proof: We write

$$\begin{aligned}
\sum_{j=m}^{n} a_j \cdot b_j &= \sum_{j=m}^{n} (A_j - A_{j-1}) \cdot b_j \\
&= \sum_{j=m}^{n} A_j \cdot b_j - \sum_{j=m}^{n} A_{j-1} \cdot b_j \\
&= \sum_{j=m}^{n} A_j \cdot b_j - \sum_{j=m-1}^{n-1} A_j \cdot b_{j+1} \\
&= \sum_{j=m}^{n-1} A_j \cdot (b_j - b_{j+1}) + A_n \cdot b_n - A_{m-1} \cdot b_m \, .
\end{aligned}$$

This is what we wished to prove. □

Now we apply summation by parts to prove a convergence test due to Niels Henrik Abel (1802–1829).

Theorem (Abel's Convergence Test)

Consider the series

$$\sum_{j=0}^{\infty} a_j \cdot b_j \, .$$

Suppose that

1. *The partial sums* $A_N = \sum_{j=0}^{N} a_j$ *form a bounded sequence;*
2. $b_0 \geq b_1 \geq b_2 \geq \ldots$;
3. $\lim_{j \to \infty} b_j = 0$.

Then the original series

$$\sum_{j=0}^{\infty} a_j \cdot b_j$$

converges.

Proof: Suppose that the partial sums A_N are bounded in absolute value by a number K. Pick $\epsilon > 0$ and choose an integer N so large that $b_N < \epsilon/(2K)$. For $N < m \leq n < \infty$ we use the partial summation formula to write

$$\begin{aligned}\left|\sum_{j=m}^{n} a_j \cdot b_j\right| &= \left|A_n \cdot b_n - A_{m-1} \cdot b_m + \sum_{j=m}^{n-1} A_j \cdot (b_j - b_{j+1})\right| \\ &\leq K \cdot |b_n| + K \cdot |b_m| + K \cdot \sum_{j=m}^{n-1} |b_j - b_{j+1}|\,.\end{aligned}$$

Now we take advantage of the facts that $b_j \geq 0$ for all j and that $b_j \geq b_{j+1}$ for all j to estimate the last expression by

$$K \cdot \left[b_n + b_m + \sum_{j=m}^{n-1} (b_j - b_{j+1}) \right] .$$

[Notice that the expressions $b_j - b_{j+1}, b_m$, and b_n are all nonnegative.] Now the sum collapses and the last line is estimated by

$$K \cdot [b_n + b_m - b_n + b_m] = 2 \cdot K \cdot b_m .$$

By our choice of N the right side is smaller than ϵ. Thus our series satisfies the Cauchy criterion and therefore converges. $\square$

Example (The Alternating Series Test)

As a first application of Abel's convergence test, we examine alternating series. Consider a series of the form

$$\sum_{j=1}^{\infty}(-1)^j \cdot b_j\,, \tag{3.32.1}$$

with $b_1 \geq b_2 \geq b_3 \geq \cdots \geq 0$ and $b_j \to 0$ as $j \to \infty$. We set $a_j = (-1)^j$ and apply Abel's test. We see immediately that all partial sums A_N are either -1 or 0. In particular, this sequence of partial sums is bounded. And the b_js are decreasing and tending to zero. By Abel's convergence test, the alternating series (3.32.1) converges. □

Proposition

Let $b_1 \geq b_2 \geq \ldots$ and assume that $b_j \to 0$. Consider the alternating series $\sum_{j=1}^{\infty}(-1)^j b_j$ as in the last example. It is convergent: let S be its sum. Then the partial sums S_N satisfy $|S - S_N| \leq b_{N+1}$.

Proof: Observe that

$$|S - S_N| = |b_{N+1} - b_{N+2} + b_{N+3} - + \ldots| \ .$$

But

$$\begin{aligned} b_{N+2} - b_{N+3} + - \ldots &\leq b_{N+2} + (-b_{N+3} + b_{N+3}) \\ &\quad + (-b_{N+5} + b_{N+5}) + \ldots \\ &= b_{N+2} \end{aligned}$$

and

$$\begin{aligned} b_{N+2} - b_{N+3} + - \ldots &\geq (b_{N+2} - b_{N+2}) + (b_{N+4} - b_{N+4}) + \ldots \\ &= 0\,. \end{aligned}$$

It follows that

$$|S - S_N| \leq |b_{N+1}|$$

as claimed. □

Example

Consider the series

$$\sum_{j=1}^{\infty}(-1)^j\frac{1}{j}\,.$$

Then the partial sum $S_{100} = -.688172$ is within 0.01 (in fact within 1/101) of the full sum S and the partial sum $S_{10000} = -.6930501$ is within 0.0001 (in fact within 1/10001) of the sum S. □