# Math 4111
# October 7, 2020 Lecture

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Figure: This is your instructor.

# 3.2 Elementary Convergence Tests

Observe that a series may converge because its terms are nonnegative and diminish in size fairly rapidly (thus causing its partial sums to grow slowly) or it may converge because of cancellation among the terms. The tests which measure the first type of convergence are the most obvious and these are the "elementary" ones that we discuss in the present section.

## Proposition (The Comparison Test)

*Suppose that $\sum_{j=1}^{\infty} a_j$ is a convergent series of nonnegative terms. If $\{b_j\}$ are real or complex numbers and if $|b_j| \leq a_j$ for every $j$ then the series $\sum_{j=1}^{\infty} b_j$ converges.*

**Proof:** Because $\sum_{j=1}^{\infty} a_j$ converges, it satisfies the Cauchy criterion for series. Hence, given $\epsilon > 0$, there is an $N$ so large that if $n \geq m > N$, then $|\sum_{j=m}^{n} a_j| < \epsilon$. But then

$$\left| \sum_{j=m}^{n} b_j \right| \leq \sum_{j=m}^{n} |b_j| \leq \sum_{j=m}^{n} a_j < \epsilon .$$

It follows that the series $\sum_{j=1}^{\infty} b_j$ satisfies the Cauchy criterion for series. Therefore it converges. □

Note that since convergence of a series is determined by its tail terms, we have the same conclusion if the inequality $|b_j| \leq a_j$ holds for all sufficiently large $j$.

As a special case of the preceding proposition, we have:

## Corollary

*If $\sum_{j=1}^{\infty} a_j$ is a convergent series of nonnegative terms and $0 \leq b_j \leq a_j$ for every $j$, then the series $\sum_{j=1}^{\infty} b_j$ converges.*

## Example

Determine if the series $\sum_{j=1}^{\infty} 2^{-j} \sin j$ converges.

Solution. Note that $\sum_{j=1}^{\infty} 2^{-j}$ converges and $|2^{-j} \sin j| \leq 2^{-j}$ for all $j$. By the Comparison Test, we see that $\sum_{j=1}^{\infty} 2^{-j} \sin j$ converges. □

# The Cauchy Condensation Test

The following result provides a useful tool to determine whether a series with nonnegative and decreasing terms converges. It suffices to examine a series only involving terms with index a power of 2.

## Theorem (The Cauchy Condensation Test)

*Assume that* $a_1 \geq a_2 \geq \cdots \geq a_j \geq \cdots \geq 0$. *The series*

$$\sum_{j=1}^{\infty} a_j$$

*converges if and only if the series*

$$\sum_{k=1}^{\infty} 2^k \cdot a_{2^k}$$

*converges.*

**Proof:** First assume that the series $\sum_{j=1}^{\infty} a_j$ converges. Notice that, for each $k \geq 1$,

$$\begin{aligned} 2^{k-1} \cdot a_{2^k} &= \underbrace{a_{2^k} + a_{2^k} + \cdots + a_{2^k}}_{2^{k-1} \text{ times}} \\ &\leq a_{2^{k-1}+1} + a_{2^{k-1}+2} + \cdots + a_{2^k}. \\ &= \sum_{m=2^{k-1}+1}^{2^k} a_m \end{aligned}$$

Therefore

$$\sum_{k=1}^{N} 2^{k-1} \cdot a_{2^k} \leq \sum_{k=1}^{N} \sum_{m=2^{k-1}+1}^{2^k} a_m = \sum_{m=2}^{2^N} a_m \,.$$

Since the partial sums on the right are bounded (because $\sum_{j=1}^{\infty} a_j$ converges), so are the partial sums on the left. It follows that the series

$$\sum_{k=1}^{\infty} 2^k \cdot a_{2^k}$$

converges.

For the converse, assume that the series

$$\sum_{k=1}^{\infty} 2^k \cdot a_{2^k} \tag{3.13.1}$$

converges.

Observe that, for $k \geq 1$,

$$\begin{aligned}\sum_{m=2^{k-1}+1}^{2^k} a_m &= a_{2^{k-1}+1} + a_{2^{k-1}+2} + \cdots + a_{2^k} \\ &\le \underbrace{a_{2^{k-1}} + a_{2^{k-1}} + \cdots + a_{2^{k-1}}}_{2^{k-1} \text{ times}} \\ &= 2^{k-1} \cdot a_{2^{k-1}} .\end{aligned}$$

It follows that $$\sum_{m=2}^{2^N} a_m = \sum_{k=1}^{N} \sum_{m=2^{k-1}+1}^{2^k} a_m \le \sum_{k=1}^{N} 2^{k-1} a_{2^{k-1}}.$$

By the hypothesis that the series (3.13.1) converges, the partial sums on the right must be bounded. But then the partial sums on the left are bounded as well. Since the summands $a_j$ are nonnegative, the series on the left converges. □

## Example

We apply the Cauchy condensation test to the harmonic series

$$\sum_{j=1}^{\infty} \frac{1}{j}\,.$$

It leads us to examine the series

$$\sum_{k=1}^{\infty} 2^k \cdot \frac{1}{2^k} = \sum_{k=1}^{\infty} 1\,.$$

Since the latter series diverges, the harmonic series diverges as well. □

# Geometric Series Test

## Proposition (Geometric Series Test)

*Let $\alpha$ be a complex number. The geometric series*

$$\sum_{j=0}^{\infty} \alpha^j$$

*converges if and only if $|\alpha| < 1$. In this circumstance, the sum of the series (that is, the limit of the partial sums) is $1/(1-\alpha)$.*

**Proof:** Let $S_N$ denote the $N$th partial sum of the geometric series. Then

$$\begin{aligned} \alpha \cdot S_N &= \alpha(1 + \alpha + \alpha^2 + \cdots + \alpha^N) \\ &= \alpha + \alpha^2 + \cdots + \alpha^{N+1}. \end{aligned}$$

It follows that $\alpha \cdot S_N$ and $S_N$ are nearly the same: in fact

$$\alpha \cdot S_N + 1 - \alpha^{N+1} = S_N \,.$$

Solving this equation for the quantity $S_N$ yields

$$S_N = \frac{1 - \alpha^{N+1}}{1 - \alpha}, \quad \text{where} \quad \alpha \neq 1.$$

If $|\alpha| < 1$ then $\alpha^{N+1} \to 0$, hence the sequence of partial sums tends to the limit $1/(1-\alpha)$. If $|\alpha| > 1$ then $\alpha^{N+1}$ diverges, hence the sequence of partial sums diverges. This completes the proof for $|\alpha| \neq 1$. But the divergence in case $|\alpha| = 1$ follows because the summands will not tend to zero. □

## Example

The series $\sum_{j=0}^{\infty} 3^{-j}$ is a geometric series. Writing it as $\sum_{j=0}^{\infty} \left(\frac{1}{3}\right)^j$, we see that the sum is $\frac{1}{1-1/3} = \frac{3}{2}$.

The sum of the series

$$\sum_{j=2}^{\infty} \left(\frac{3}{4}\right)^j = \left(\frac{3}{4}\right)^2 \sum_{j=0}^{\infty} \left(\frac{3}{4}\right)^j$$

is $\frac{9}{16} \cdot \frac{1}{1-3/4} = \frac{9}{16} \cdot 4 = \frac{9}{4}$.

More generally, when $|\alpha| < 1$, the sum of the geometric series $\sum_{j=m}^{\infty} c\,\alpha^j$ is $\frac{c\alpha^m}{1-\alpha}$.

For a real number $p$, the series $\sum_{j=1}^{\infty} \frac{1}{j^p}$ is called the $p$-series.

## Theorem (The $p$-Series Test)

*Let $p$ be a real number. The series* $\sum_{j=1}^{\infty} \frac{1}{j^p}$ *converges if* $p > 1$ *and diverges otherwise.*

**Proof:** Clearly, the series diverges when $p \leq 0$.
When $p > 0$, we can apply the Cauchy Condensation Test. This leads us to examine the series

$$\sum_{k=1}^{\infty} 2^k \cdot 2^{-kp} = \sum_{k=1}^{\infty} \left(2^{1-p}\right)^k,$$

a geometric series with the common ratio of a term to its previous term being $\alpha = 2^{1-p}$. When $p > 1$ then $|\alpha| < 1$ so the series converges. Otherwise, it diverges. □

# The Limit Comparison Test

In some situations, for two series with positive terms, direct comparison may be inconvenient, but the limit of the ratios of corresponding terms may be used instead.

## Theorem (The Limit Comparison Test)

*Let* $\sum_{n=1}^{\infty} a_n$ *and* $\sum_{n=1}^{\infty} b_n$ *be series with positive terms.*
*(a). If* $\sum_{n=1}^{\infty} b_n$ *converges and* $\lim_{n\to\infty} \frac{a_n}{b_n} = M < \infty$, *then* $\sum_{n=1}^{\infty} a_n$ *converges.*
*(b). If* $\sum_{n=1}^{\infty} b_n$ *diverges and* $\lim_{n\to\infty} \frac{a_n}{b_n} = \ell > 0$, *then* $\sum_{n=1}^{\infty} a_n$ *diverges.*

**Proof.** To prove (a), assume that $\sum_{n=1}^{\infty} b_n$ converges and $\lim_{n\to\infty} \frac{a_n}{b_n} = M < \infty$. Then for all sufficiently large $n$, we have $\frac{a_n}{b_n} < M + 1$, and hence, $a_n < (M+1)b_n$. Obviously $\sum_{n=1}^{\infty}(M+1)b_n$ converges. By the Comparison Test, we see that $\sum a_n$ converges. The proof of (b) is similar, and is omitted. □

## Example

Determine convergence or divergence of each series:

(a) $\displaystyle\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{5n^2 - 4n + 3}{n^4 + 8n^2 + 10}$

(b) $\displaystyle\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{n^3\sqrt{n} + n + 3}{3n^4 + 8n^2 + 10}$

**Solution.** Note that $\sum_{n=1}^{\infty} 1/n^2$ converges by the $p$-Series Test, and $\lim \dfrac{a_n}{1/n^2} = 5 < \infty$. By the Limit Comparison Test, $\sum_{n=1}^{\infty} a_n$ converges.

By the $p$-Series Test, $\sum_{n=1}^{\infty} \dfrac{1}{n^{1/2}}$ diverges. Note that $\lim_{n\to\infty} \dfrac{b_n}{1/n^{1/2}} = 1/3 > 0$. By the Limit Comparison Test, we see that $\sum_{n=1}^{\infty} b_n$ diverges. □