Math 4111
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Figure: This is your instructor.

Convergence of Series

In this section we will use standard summation notation:

$$\sum_{j=m}^{n} a_j \equiv a_m + a_{m+1} + \cdots + a_n \,.$$

A series is an infinite sum. One of the most effective ways to handle an infinite process in mathematics is with a limit. This consideration leads to the following definition:

Definition

The formal expression

$$\sum_{j=1}^{\infty} a_j \, ,$$

where the a_js are real or complex numbers, is called a *series*. For $N = 1, 2, 3, \ldots$, the expression

$$S_N = \sum_{j=1}^{N} a_j = a_1 + a_2 + \ldots a_N$$

is called the Nth *partial sum* of the series.

In case

$$\lim_{N\to\infty} S_N$$

exists and is finite we say that the series *converges*. The limit of the partial sums is called the *sum* of the series. If the series does not converge, then we say that the series *diverges*.

Notice that the question of convergence of a series, which should be thought of as an *addition process*, reduces to a question about the *sequence* of partial sums.

Example

Consider the series

$$\sum_{j=1}^{\infty} 2^{-j}.$$

The Nth partial sum for this series is

$$S_N = 2^{-1} + 2^{-2} + \cdots + 2^{-N}.$$

In order to determine whether the sequence $\{S_N\}$ has a limit, we rewrite S_N as

$$\begin{aligned} S_N &= \left(2^{-0} - 2^{-1}\right) + \left(2^{-1} - 2^{-2}\right) + \ldots \\ &\qquad \left(2^{-N+1} - 2^{-N}\right). \end{aligned}$$

The expression on the right of the last equation telescopes (i.e., successive pairs of terms cancel) and we find that

$$S_N = 2^{-0} - 2^{-N}.$$

Thus

$$\lim_{N\to\infty} S_N = 2^{-0} = 1.$$

We conclude that the series converges. □

Example

Let us examine the series

$$\sum_{j=1}^{\infty} \frac{1}{j}$$

for convergence or divergence. (This series is commonly called the *harmonic series* because it describes the harmonics in music.) Now

$$
\begin{aligned}
S_1 &= 1 = \frac{2}{2} \\
S_2 &= 1 + \frac{1}{2} = \frac{3}{2} \\
S_4 &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) \\
&\geq 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) \geq 1 + \frac{1}{2} + \frac{1}{2} = \frac{4}{2} \\
S_8 &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) \\
&\geq 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) \\
&= \frac{5}{2}.
\end{aligned}
$$

In general this argument shows that

$$S_{2^k} \geq \frac{k+2}{2}.$$

The sequence of S_Ns is increasing since the series contains only positive terms. The fact that the partial sums $S_1, S_2, S_4, S_8, \ldots$ increases without bound shows that the entire sequence of partial sums must increase without bound. We conclude that the series diverges. □

Just as with sequences, we have a Cauchy criterion for series:

Proposition

The series $\sum_{j=1}^{\infty} a_j$ converges if and only if, for every $\epsilon > 0$, there is an integer $N \geq 1$ such that, if $n \geq m > N$, then

$$\left| \sum_{j=m}^{n} a_j \right| < \epsilon. \qquad (*)$$

The condition $(*)$ is called the Cauchy criterion for series.

Proof: Suppose that the Cauchy criterion holds. Pick $\epsilon > 0$ and choose N so large that $(*)$ holds. If $n \geq m > N$, then

$$|S_n - S_m| = \left| \sum_{j=m+1}^{n} a_j \right| < \epsilon$$

by hypothesis. Thus the sequence $\{S_N\}$ is Cauchy in the sense discussed for sequences in an earlier lecture. We conclude that the sequence $\{S_N\}$ converges; by definition, therefore, the series converges.

Conversely, if the series converges then, by definition, the sequence $\{S_N\}$ of partial sums converges. In particular, the sequence $\{S_N\}$ must be Cauchy. Thus, for any $\epsilon > 0$, there is a number $N > 0$ such that if $n \geq m > N$ then

$$|S_n - S_m| < \epsilon \, .$$

This just says that

$$\left| \sum_{j=m+1}^{n} a_j \right| < \epsilon \, ,$$

and this last inequality is the Cauchy criterion for series. $\square$

Example

Let us use the Cauchy criterion to verify that the series

$$\sum_{j=1}^{\infty} \frac{1}{j \cdot (j+1)}$$

converges.

Notice that, if $n \geq m > 1$, then

$$\left| \sum_{j=m}^{n} \frac{1}{j \cdot (j+1)} \right| = \left(\frac{1}{m} - \frac{1}{m+1} \right) + \left(\frac{1}{m+1} - \frac{1}{m+2} \right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1} \right).$$

The sum on the right plainly telescopes and we have

$$\left| \sum_{j=m}^{n} \frac{1}{j \cdot (j+1)} \right| = \frac{1}{m} - \frac{1}{n+1}.$$

Let $\epsilon > 0$. Let us choose N to be the next integer after $1/\epsilon$. Then, for $n \geq m > N$, we may conclude that

$$\left| \sum_{j=m}^{n} \frac{1}{j \cdot (j+1)} \right| = \frac{1}{m} - \frac{1}{n+1} < \frac{1}{m} < \frac{1}{N} < \epsilon.$$

This is the desired conclusion. □

The next result gives a necessary condition for a series to converge. It is a useful device for detecting divergent series, although it can never tell us that a series converges.

Proposition (The Zero Test)

If the series

$$\sum_{j=1}^{\infty} a_j$$

converges then the terms a_j tend to zero as $j \to \infty$.

Proof: Since we are assuming that the series converges, then it must satisfy the Cauchy criterion. Let $\epsilon > 0$. Then there is an integer $N \geq 1$ such that, if $n \geq m > N$, then

$$\left| \sum_{j=m}^{n} a_j \right| < \epsilon \,. \qquad (**)$$

We take $n = m$ and $m > N$. Then $(**)$ becomes

$$|a_m| < \epsilon \,.$$

But this is precisely the conclusion that we desire. $\square$

Example

The series $\sum_{j=1}^{\infty}(-1)^j$ must diverge, *even though its terms appear to be cancelling each other out.* The reason is that the summands do not tend to zero; hence the preceding proposition applies.

Write out several partial sums of this series to see more explicitly that the partial sums are -1, $+1$, -1, $+1$, ... and hence that the series diverges. □

We conclude this section with a necessary and sufficient condition for convergence of a series of nonnegative terms. As with some of our other results on series, it amounts to little more than a restatement of a result on sequences.

Proposition

A series

$$\sum_{j=1}^{\infty} a_j$$

with all $a_j \geq 0$ *is convergent if and only if the sequence of partial sums is bounded.*

Proof: Notice that, because the summands are nonnegative, we have

$$S_1 = a_1 \leq a_1 + a_2 = S_2\,,$$

$$S_2 = a_1 + a_2 \leq a_1 + a_2 + a_3 = S_3\,,$$

and in general

$$S_N \leq S_N + a_{N+1} = S_{N+1}\,.$$

Thus the sequence $\{S_N\}$ of partial sums forms a increasing sequence. We know that such a sequence is convergent to a finite limit if and only if it is bounded above (see our earlier discussion). This completes the proof. □

Example

The series $\sum_{j=1}^{\infty} 1$ is divergent since the summands are nonnegative and the sequence of partial sums $\{S_N\} = \{N\}$ is unbounded.

Referring back to to our example on harmonic series, we see that the series $\sum_{j=1}^{\infty} \frac{1}{j}$ diverges because its partial sums are unbounded.

We see from the first example that the series $\sum_{j=1}^{\infty} 2^{-j}$ converges because its partial sums are all bounded above by 1.

□

It is frequently convenient to begin a series with summation at $j = 0$ or some other term instead of $j = 1$. All of our convergence results still apply to such a series because of the Cauchy criterion. In other words, the convergence or divergence of a series will depend only on the behavior of its "tail."