Math 4111
October 2, 2020 Lecture

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September 2, 2020



Figure: This is your instructor.

Some Special Sequences

We often obtain information about a new sequence by comparison with a sequence that we already know. Thus it is well to have a catalogue of fundamental sequences which provide a basis for comparison.

Example

Fix a real number a. The sequence $\{a^j\}$ is called a *power sequence*. If $-1 < a < 1$ then the sequence converges to 0. If $a = 1$ then the sequence is a constant sequence and converges to 1. If $a > 1$ then the sequence diverges to $+\infty$. Finally, if $a \leq -1$ then the sequence diverges. □

Recall that, in the first lecture, we discussed the existence of nth roots of positive real numbers. If $\alpha > 0$, $m \in \mathbb{Z}$, and $n \in \mathbb{N}$ then we may define

$$\alpha^{m/n} = (\alpha^m)^{1/n}.$$

Thus we may talk about rational powers of a positive number.

Next, if $\beta \in \mathbb{R}$ then we may define

$$\alpha^\beta = \sup\{\alpha^q : q \in \mathbb{Q}, q < \beta\}.$$

Thus we can define *any real power* of a positive real number. The exercises ask you to verify several basic properties of these exponentials.

Lemma

If $\alpha > 1$ is a real number and $\beta > 0$ then $\alpha^\beta > 1$.

Proof: Let q be a positive rational number which is less than β. Suppose that $q = m/n$, with m, n integers. It is obvious that $\alpha^m > 1$ and hence that $(\alpha^m)^{1/n} > 1$. Since α^β majorizes this last quantity, we are done. □

Example

Fix a real number α and consider the sequence $\{j^\alpha\}$. If $\alpha > 0$ then it is easy to see that $j^\alpha \to +\infty$: to verify this assertion fix $M > 0$ and take the number N to be the first integer after $M^{1/\alpha}$. If $\alpha = 0$ then j^α is a constant sequence, identically equal to 1. If $\alpha < 0$ then $j^\alpha = 1/j^{-\alpha}$. The denominator of this last expression tends to $+\infty$ hence the sequence j^α tends to 0. □

Example

The sequence $\{j^{1/j}\}$ converges to 1. In fact, consider the expressions $\alpha_j = j^{1/j} - 1 > 0$. We have that

$$j = (\alpha_j + 1)^j \geq \frac{j(j-1)}{2}(\alpha_j)^2\,,$$

(the latter being just one term from the binomial expansion). Thus

$$0 < \alpha_j \leq \sqrt{2/(j-1)}$$

as long as $j \geq 2$. It follows that $\alpha_j \to 0$ or $j^{1/j} \to 1$. □

Example

Let α be a positive real number. Then the sequence $\alpha^{1/j}$ converges to 1. To see this, first note that the case $\alpha = 1$ is trivial, and the case $\alpha > 1$ implies the case $\alpha < 1$ (by taking reciprocals). So we concentrate on $\alpha > 1$. But then we have

$$1 < \alpha^{1/j} < j^{1/j}$$

when $j > \alpha$. Since $j^{1/j}$ tends to 1, an earlier proposition applies and the proof is complete. □

Example

Let $\lambda > 1$ and let α be real. Then the sequence

$$\left\{\frac{j^\alpha}{\lambda^j}\right\}_{j=1}^{\infty}$$

converges to 0.

To see this, fix an integer $k > \alpha$ and consider $j > 2k$. [Notice that k is fixed once and for all but j will be allowed to tend to $+\infty$ at the appropriate moment.] Writing $\lambda = 1 + \mu, \mu > 0$, we have that

$$\lambda^j = (1+\mu)^j > \frac{j(j-1)(j-2)\cdots(j-k+1)}{k(k-1)(k-2)\cdots 2\cdot 1}\cdot 1^{j-k}\cdot \mu^k\,.$$

Of course this comes from picking out the kth term of the binomial expansion for $(1+\mu)^j$. Notice that, since $j > 2k$, then each of the expressions $j, (j-1), \ldots (j-k+1)$ in the numerator on the right exceeds $j/2$. Thus

$$\lambda^j > \frac{j^k}{2^k \cdot k!} \cdot \mu^k$$

and

$$0 < \frac{j^\alpha}{\lambda^j} < j^\alpha \cdot \frac{2^k \cdot k!}{j^k \cdot \mu^k} = \frac{j^{\alpha-k} \cdot 2^k \cdot k!}{\mu^k}.$$

Since $\alpha - k < 0$, the right side tends to 0 as $j \to \infty$. □

Example

The sequence

$$\left\{\left(1+\frac{1}{j}\right)^j\right\}$$

converges. In fact it is increasing and bounded above. Use the Binomial Expansion to prove this assertion. The limit of the sequence is the number that we shall later call e (in honor of Leonhard Euler, 1707–1783, who first studied it in detail). We shall study this sequence in detail later in the book. □

Example

The sequence

$$\left(1 - \frac{1}{j}\right)^j$$

converges to $1/e$, where the definition of e is given in the last example. More generally, the sequence

$$\left(1 + \frac{x}{j}\right)^j$$

converges to e^x (here e^x is defined as in the earlier discussion).

□