Math 4111
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Figure: This is your instructor.

2.3 Lim sup and lim inf

Exercises: 1–13.

Convergent sequences are useful objects, but the unfortunate truth is that most sequences do not converge.

Nevertheless, we would like to have a language for discussing the asymptotic behavior of *any* real sequence (a_n) as $n \to \infty$, such as the largest or smallest possible subsequence limit. That is the purpose of the concepts of "limit superior" (or "upper limit") and "limit inferior" (or "lower limit").

It is common to use the notation "lim sup" and "lim inf" to describe these terms.

Definition of lim inf (lower limit)

Definition

Let $\{a_j\}$ be a real sequence that is bounded below. For each j, let

$$A_j = \inf\{a_j, a_{j+1}, a_{j+2}, \dots\}.$$

Then $\{A_j\}$ is an increasing sequence (since, as j becomes large, we are taking the infimum of a smaller set of numbers), so it has a limit (either a finite limit or $+\infty$). We define the *limit infimum* of $\{a_j\}$ to be

$$\liminf_{j\to\infty} a_j = \lim_{j\to\infty} A_j\,.$$

It is common to refer to this number as the *lower limit* of the sequence. By convention, we also say that the lim inf of $\{a_j\}$ is $-\infty$ when $\{a_j\}$ is not bounded below.
The lower limit of $\{a_j\}$ is also denoted $\underline{\lim}_{j\to\infty} a_j$.

Definition of lim sup (upper limit)

Definition

Let $\{a_j\}$ be a real sequence that is bounded above. For each $j \in \mathbb{N}$, Let

$$B_j = \sup\{a_j, a_{j+1}, a_{j+2}, \dots\}.$$

Then $\{B_j\}$ is a decreasing sequence (since, as j becomes large, we are taking the supremum of a smaller set of numbers), so it has a limit (either a finite limit or $-\infty$). We define the *limit supremum* of $\{a_j\}$ to be

$$\limsup a_j = \lim_{j\to\infty} B_j.$$

It is common to refer to this number as the *upper limit* of the sequence. By convention, we also say that the lim sup of $\{a_j\}$ is $+\infty$ when $\{a_j\}$ is not bounded above.

The upper limit of $\{a_j\}$ is also denoted $\overline{\lim}_{j\to\infty} a_j$.

lim sup is the greatest possible subsequence limit

Notice that the lim sup or lim inf of a sequence can be $\pm\infty$.

Remark

What is the intuitive content of this definition? For each j, A_j picks out the greatest lower bound of the sequence in the j^{th} position or later. So the sequence $\{A_j\}$ should tend to the smallest *possible limit of any subsequence of $\{a_j\}$.*

Likewise, for each j, B_j picks out the least upper bound of the sequence in the j^{th} position or later. So the sequence $\{B_j\}$ should tend to the greatest *possible limit of any subsequence of $\{a_j\}$. We shall make these remarks more precise in the proposition below.*

A simple example

Notice that it is implicit in the definition that *every* real sequence has a limit supremum and a limit infimum.

Example

Consider the sequence $\{(-1)^j\}$. Clearly, this sequence does not converge. Calculate its $\liminf$ and $\limsup$.

Solution. Referring to the definition, we have that $A_j = -1$ and $B_j = 1$ for every j. So
$\liminf_{j\to\infty}(-1)^j = \lim_{j\to\infty}(-1) = -1.$, and
$\limsup_{j\to\infty}(-1)^j = \lim_{j\to\infty} 1 = 1.$
As we predicted in the remark, the $\liminf$ is the least subsequece limit, and the $\limsup$ is the greatest subsequence limit. □

Now let us prove the characterizing property of lim sup and lim inf to which we have been alluding.

Proposition

Let $\{a_j\}$ be a sequence of real numbers. Let $\beta = \limsup_{j\to\infty} a_j$ and $\alpha = \liminf_{j\to\infty} a_j$. If $\{a_{j_\ell}\}$ is any subsequence of the given sequence then

$$\alpha \leq \liminf_{\ell\to\infty} a_{j_\ell} \leq \limsup_{\ell\to\infty} a_{j_\ell} \leq \beta\,.$$

Moreover, there is a subsequence $\{a_{j_k}\}$ such that

$$\lim_{k\to\infty} a_{j_k} = \alpha$$

and another sequence $\{a_{j_m}\}$ such that

$$\lim_{k\to\infty} a_{j_m} = \beta\,.$$

Proof: For simplicity in this proof we assume that the lim sup and lim inf are finite. The case of infinite lim sups and lim infs is treated in the exercises.
We begin by considering the lim inf. There is a $j_1 \geq 1$ such that $|A_1 - a_{j_1}| < 2^{-1}$. Next, we choose j_2, necessarily greater than j_1, such that $|a_{j_2} - A_{j_1+1}| < 2^{-2}$. Continuing in this fashion, we select $j_k > j_{k-1}$ such that $|a_{j_k} - A_{j_{k-1}+1}| < 2^{-k}$, etc.

Clearly, $A_{j_k+1} \to \alpha = \liminf_{j\to\infty} a_j$. Now fix $\epsilon > 0$. If N is an integer so large that $k > N$ implies that $|A_{j_{k-1}+1} - \alpha| < \epsilon/2$ and also that $2^{-N} < \epsilon/2$ then, for such k, we have

$$\begin{aligned} |a_{j_k} - \alpha| &\leq \left|a_{j_k} - A_{j_{k-1}+1}\right| + |A_{j_{k-1}+1} - \alpha| \\ &< 2^{-k} + \frac{\epsilon}{2} \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon\,. \end{aligned}$$

Thus the subsequence $\{a_{j_k}\}$ converges to α, the $\liminf$ of the given sequence. A similar construction gives a (different) subsequence $\{a_{n_k}\}$ converging to β, the $\limsup$ of the given sequence.

Now let $\{a_{j_\ell}\}$ be *any* subsequence of the sequence $\{a_j\}$. Let β^* be the lim sup of this subsequence. Then, by the first part of the proof, there is a subsequence $\{a_{j_{\ell_m}}\}$ such that

$$\lim_{m\to\infty} a_{j_{\ell_m}} = \beta^* .$$

But $a_{j_{\ell_m}} \le B_{j_{\ell_m}}$ by the very definition of the Bs. Thus

$$\beta^* = \lim_{m\to\infty} a_{j_{\ell_m}} \le \lim_{m\to\infty} B_{j_{\ell_m}} = \beta$$

or

$$\limsup_{\ell\to\infty} a_{j_\ell} \le \beta ,$$

as claimed. A similar argument shows that

$$\liminf_{l\to\infty} a_{j_l} \ge \alpha .$$

This completes the proof of the proposition. □

Corollary

If $\{a_j\}$ is a sequence and $\{a_{j_k}\}$ is a convergent subsequence then

$$\liminf_{j\to\infty} a_j \leq \lim_{k\to\infty} a_{j_k} \leq \limsup_{j\to\infty} a_j \,.$$

Example

Consider the sequence

$$a_j = \frac{(-1)^j j^2}{j^2 + j}\,.$$

It is helpful to rewrite this sequence as

$$a_j = (-1)^j \cdot \left(1 - \frac{j}{j^2 + j}\right) = (-1)^j \cdot \left(1 - \frac{1}{j+1}\right)\,.$$

Then, looking at the terms of even index, it is easy to see that the lim sup of this sequence is $+1$. And, looking at the terms of odd index, it is easy to see that the lim inf of this sequence is -1.

Every convergent subsequence of $\{a_j\}$ will have limit lying between -1 and $+1$. □

A fact about lim sup and lim inf

The following fact about sequences is analogous to an earlier result on the supremum and infimum of a nonempty set.

Proposition

Let $\{a_j\}$ *be a real sequence and set* $\liminf a_j = \alpha$ *and* $\limsup a_j = \beta$. *Assume that* α, β *are finite real numbers. Let* $\epsilon > 0$. *Then there are arbitrarily large* j *such that* $a_j > \beta - \epsilon$. *Also there are arbitrarily large* k *such that* $a_k < \alpha + \epsilon$.

Proof. let $\epsilon > 0$. As proved earlier, there is a subsequence $\{a_{j_k}\}$ that converges to β. Hence, there is a positive integer N such that $|a_{j_k} - \beta| < \epsilon$ (namely, $\beta - \epsilon < a_{j_k} < \beta + \epsilon$) for all $k > N$. Thus there are arbitrarily large values of j such as $a_j > \beta - \epsilon$.
The proof concerning α is similar. □

Example

Consider the sequence $\{a_j\} = \{\frac{(-1)^j j^2}{j^2+j}\}$ in the last example. Let $\epsilon > 0$. Choose j even so that $j > 1/\epsilon$. Then

$$\frac{j^2}{j^2+j} = 1 - \frac{1}{j+1} > 1 - \frac{1}{j} > 1 - \epsilon\,.$$

Choose j odd so that $j > 1/\epsilon$. Then

$$-\frac{j^2}{j^2+j} < -1 + \epsilon\,.$$

Concluding example and exercise

Example

Determine all possible limits of subsequences of the sequence $(\cos n)$.

Solution. For any integer n, let $n \bmod 2\pi$ denote $n - \lfloor \frac{n}{2\pi} \rfloor 2\pi$. As shown in earlier exercises, the set $\{n \bmod 2\pi : n \in \mathbb{N}\}$ is dense in $[0, 2\pi]$. By the Mean Value Theorem, for any two real numbers a and b, $|\cos b - \cos a| = |(\sin c)(b - a)| \leq |b - a|$, for some number c between a and b. Hence, $\{\cos n : n \in \mathbb{N}\}$ $= \{cos(n \bmod 2\pi) : n \in \mathbb{N}\}$ is dense in $[-1, 1]$. Therefore, $\liminf_{n\to\infty} \cos n = -1$, $\limsup_{n\to\infty} \cos n = 1$, and for every number $x \in [-1, 1]$, there is a subsequence of $(\cos n)$ converging to x. □

Exercise: Show that for every real number r, there is a subsequence of $(\tan n)$ that converges to r.