Math 4111
September 28, 2020 Lecture

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September 26, 2020



Figure: This is your instructor.

2.2 Subsequences

Homework: 1, 2, 4, 8.

Given a sequence $\{a_n\}$. For **strictly increasing positive integers** $n_1 < n_2 < n_3 < \cdots$, the sequence

$$a_{n_1}, a_{n_2}, a_{n_3}, \ldots$$

is called a *subsequence* of the sequence $\{a_n\}$, denoted $\{a_{n_k}\}_{k=1}^{\infty}$ or $\{a_{n_k}\}$. So a subsequence contains a subcollection of terms of the original sequence *in the same order*.

Caution: $a_{n_1}, a_{n_1}, a_{n_1}, \ldots$ cannot be called a subsequence of $\{a_n\}$ as the subscripts (or indices) are not strictly increasing.

Example

Consider the sequence $\{2^n\} = \{2, 4, 8, 16, \ldots\}$. Then the sequence $\{2^{2k}\} = \{4, 16, 64, \ldots\}$ is a subsequence, with $n_k = 2k$. However, $\{2^{1+(k-9)^2}\}$ is not a subsequence of $\{2^n\}$ as the positive integers $1 + (k-9)^2$ are not strictly increasing as k increases.

Convergence of subsequences

Proposition

If $\{a_j\}$ is a convergent sequence with limit α, then every subsequence of $\{a_j\}$ converges to the limit α.
Conversely, if a sequence $\{b_j\}$ has the property that each of its subsequences is convergent then $\{b_j\}$ itself is convergent.

Proof. Assume that $\{a_j\}$ converges to α, and let $\{a_{j_k}\}$ be a subsequence. Let $\epsilon > 0$ and choose $N > 0$ such that $|a_j - \alpha| < \epsilon$ whenever $j > N$. Now if $k > N$ then $j_k > N$ hence $|a_{j_k} - \alpha| < \epsilon$. Therefore, by definition, the subsequence $\{a_{j_k}\}$ also converges to α.
The converse is trivial, simply because the sequence is a subsequence of itself. □

Every real sequence has a monotone subsequece

The second part of the above proposition may be strengthened as follows: If every subsequence of $\{b_j\}$ converges, then all the subsequences of $\{b_j\}$ converge to the same limit. The proof of this claim is left as an exercise.

We now show a very interesting property of sequences of real numbers (called real sequences for short). We say a term a_j of a real sequence $\{a_n\}$ is (*forward*) *dominating* if $a_j \geq a_k$ for all $k \geq j$, that is, a_j is greater than or equal to every term after it.

Theorem

Every real sequence (a_n) has a monotone subsequence.

Proof. If $\{a_n\}$ contains infinitely many dominating terms $a_{n_1}, a_{n_2}, \ldots$ with $n_1 < n_2, \ldots$, then $\{a_{n_k}\}$ is clearly a decreasing subsequence of $\{a_n\}$.

Every real sequence has a monotone subsequece

Assume that $\{a_n\}$ has *only finitely many* dominating terms. Let n_1 be a positive integer such that for every $n \geq n_1$, a_n is not dominating. Since a_{n_1} is not dominating, there is an $n_2 \in \mathbb{N}$ such that $n_1 < n_2$ and $a_{n_1} < a_{n_2}$. Since a_{n_2} is not dominating, there is an $n_3 \in \mathbb{N}$ such that $n_2 < n_3$ and $a_{n_2} < a_{n_3}$. Continuing this way, we obtain an increasing subsequence $\{a_{n_k}\}$. □

The above theorem ensures that the sequence $\{\sin n\}$ has a monotone subsequence. But finding a specific monotone subsequence is a challenge.

Since every subsequence of a bounded sequence is bounded and every bounded monotone sequence converges, we arrive at the following fundamental result (due to B. Bolzano, 1781–1848, and K. Weierstrass, 1815–1897).

Bolzano-Weierstrass Theorem

Theorem (Bolzano–Weierstrass)

Let (a_n) be a bounded sequence of real numbers. Then there is a subsequence that converges.

It is instructive to read the alternative proof of this result presented in the textbook, starting with an interval $[-M, M]$ (where $M > 0$) that contains all the terms of the sequence and pick any term a_{n_1}, then bisect the interval into two subintervals of equal lengths and from a subinterval of length M containing infinitely many terms of (a_n), take a term a_{n_2} with $n_2 > n_1$. Continue in this fashion to obtain a subsequence (a_{n_k}). Note that a_{n_k} and all subsequent terms in this subsequence are in an interval of length $4M/2^k$, so the subsequence is Cauchy, and hence, convergent.

A sequence that has a subsequence converging to any number in $[0, 1]$

Example

Construct a sequence (a_n) such that for each real number x, there is a subsequence of (a_n) that converges to x.

Solution. Since the set $\mathbb{Q}$ is countable, there is a sequence (a_n) whose terms are all the elements of $\mathbb{Q}$, with each element of $\mathbb{Q}$ occurring exactly once. Let x. Choose n_1 such that $|a_{n_1} - x| < 1$. Suppose that $n_1 < n_2 < \cdots < n_t$ have been chosen such that $|a_{n_j} - x| < 1/j$ for all $j = 1, \ldots, t$. Choose n_{t+1} such that $n_t < n_{t+1}$ and $|a_{n_{t+1}} - x| < 1/(t+1)$, which is possible as there are infinitely many rational numbers in $(x - \frac{1}{t+1}, x + \frac{1}{t+1})$. It is then clear that the subsequence (a_{n_k}) converges to x. $\square$

Bolzano-Weierstrass Theorem for complex numbers

Clearly, the sequence $(a_n) = (1 + (-1)^n) = (0, 2, 0, 2, \dots)$ diverges. A subsequence of this sequence converges if and only if it has only finitely many terms equal to 0 or it has only finitely many terms equal to 2.

Corollary (Complex Bolzano-Weierstrass Theorem)

Let $\{\alpha_j\}$ be a bounded sequence of complex numbers. Then there is a convergent subsequence.

Proof: Write $\alpha_j = a_j + ib_j$, with $a_j, b_j \in \mathbb{R}$. The fact that $\{\alpha_j\}$ is bounded implies that $\{a_j\}$ is bounded. By the Bolzano–Weierstrass theorem, there is a convergent subsequence $\{a_{j_k}\}$.
Now the sequence $\{b_{j_k}\}$ is bounded. So it has a convergent subsequence $\{b_{j_{k_l}}\}$. Then the sequence $\{\alpha_{j_{k_l}}\}$ is convergent, and is a subsequence of the original sequence $\{\alpha_j\}$. □

Divergence to $+\infty$ or $-\infty$

Definition

We say that a real sequence *diverges to* $+\infty$ if for every real number $M > 0$, there is a positive integer N such that $a_j > M$ for all $j > N$. In this case, we say that $\{a_j\}$ has limit $+\infty$ and we write $\lim_{j\to\infty} a_j = +\infty$, or $a_j \to +\infty$.

Divergence to $-\infty$ is defined similarly. Note that

$$\lim_{j\to\infty} a_j = -\infty \text{ if and only if } \lim_{j\to\infty} -a_j = +\infty.$$

For example, $\{n^3 - 20n - 100\} = \{n^3(1 - \frac{20}{n^2} - \frac{100}{n^3})\}$ diverges to $+\infty$, while from calculus, we also know that $\{\frac{2^n}{-n^5}\}$ diverges to $-\infty$. However, the sequence $\{n + (-1)^n \cdot n\}$ has no infinite limit and no finite limit, although it has a subsequence (using even subscripts only) that diverges to $+\infty$ and it also has subsequence (using odd subscripts only) that converges to 0.

Every monotone sequence has a limit, possibly $\pm\infty$

Proposition

Let $\{a_j\}$ be an increasing sequence of real numbers. Then the sequence has a limit—either a finite number or $+\infty$.
Let $\{b_j\}$ be a decreasing sequence of real numbers. Then the sequence has a limit—either a finite number or $-\infty$.

In the same spirit as the last definition, we also have the following:

Definition

If S is a set of real numbers which is *not* bounded above, we say that its supremum (or least upper bound) is $+\infty$.
If T is a set of real numbers which is *not* bounded below, then we say that its infimum (or greatest lower bound) is $-\infty$.

Negation of $lim_{n\to\infty} a_n = \alpha$

Let $\alpha \in \mathbb{C}$. It is useful to have precise understanding of the negation of "the sequence (a_n) converges to α". Note that the negation includes two possibilities, either (a_n) diverges, or (a_n) converges to a number different from α. But such formulation is not convenient to apply sometimes.

A precise and specific negation of "the sequence (a_n) converges to α" comes directly from the ϵ-N definition.

Fact

The sequence (a_n) does not converge to a number α if and only if there is a positive number $\epsilon_1 > 0$ such that for each positive integer N, there is a positive integer $n > N$ such that $|a_n - \alpha| \geq \epsilon_1$.

Negation of $lim_{n\to\infty} a_n = \alpha$; limits of subsequences

In particular, the above fact may be used to show that if a sequence (a_n) does not converge to a number α, then (a_n) has a subsequence that does not have any subsequence converging to α.

For a given sequence (a_n), how can we construct a sequence (b_n) with the property that each term a_j is the limit of a subsequence of (b_n)? Note that the set

$$S = \{a_j + 1/k : j, k \in \mathbb{N}\}$$

is countable, so there is a sequence (b_n) whose terms are all the elements of S. It is left as an exercise to verify that (b_n) has the desired property.