Math 4111
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Figure: This is your instructor.

Applying limit properties

Example

Let $a_j = \sin(2/j)/(1/j)$ and $b_j = (3j^2 - 5)/(2j^2 + j)$. Find $\lim_{j\to\infty} a_j$, $\lim_{j\to\infty} b_j$, and $\lim_{j\to\infty}(a_j + 6b_j)/(a_j b_j)$.

Solution. Recall from calculus that $\lim_{x\to 0} \frac{\sin x}{x} = 1$, where x may be replaced with any nonzero quantity approaching 0. As $2/j \to 0$, we have $\lim_{j\to\infty} \frac{\sin(2/j)}{2/j} = 1$ and hence,

$$\begin{aligned}\lim_{j\to\infty} a_j &= \lim_{j\to\infty} \sin(2/j)/(1/j) \\ &= \lim_{j\to\infty} 2\frac{\sin(2/j)}{2/j} \\ &= 2.\end{aligned}$$

Using algebra and the preceding theorem, we get $\lim_{j\to\infty} b_j = \lim_{j\to\infty} \frac{3j^2-5}{2j^2+j} = \lim_{j\to\infty} \frac{(3j^2-5)/j^2}{(2j^2+j)/j^2} = \lim_{j\to\infty} \frac{3-5/j^2}{2+1/j} = (3-0)/(2+0) = 3/2$, and

$\lim_{j\to\infty}(a_j + 6b_j)/(a_j b_j) = (2 + 6\cdot(3/2))/(2\cdot 3/2) = 11/3.$ □

Cauchy sequences

Note that trying to determine if a sequence converges using the ϵ-N definition directly is quite inconvenient, as "the limit" is often unavailable when we do not even know if the sequence converges.
Fortunately, there is a way to determine whether a sequence converges *without knowing to what limit it might converge*.

Definition (Cauchy Sequences)

Let $\{a_j\}$ be a sequence of real (or complex) numbers. We say that the sequence $\{a_j\}$ satisfies the *Cauchy criterion* (A. L. Cauchy, 1789–1857), or more briefly, the sequence is *Cauchy*, if, for each $\epsilon > 0$, there is an integer $N > 0$ such that if $j, k > N$ then $|a_j - a_k| < \epsilon$.

We may say that a sequence is Cauchy if the tail terms of the sequence are arbitrarily close to each other as long as their indices are sufficiently large.

Examples of Cauchy sequences

Example

Assume that the sequence (a_n) satisfies $|a_n - a_m| < \frac{1}{n} + \frac{1}{m}$. Show that the sequence (a_n) is Cauchy.

Proof. Let $\epsilon > 0$. Choose an integer $N > 0$ such that $N > \frac{2}{\epsilon}$. For example, we may let $N = 1 + \lfloor \frac{2}{\epsilon} \rfloor$. Then $\frac{1}{N} < \epsilon/2$ and for all integers $n, m > N$, we have $|a_n - a_m| < \frac{1}{n} + \frac{1}{m} < \frac{1}{N} + \frac{1}{N} < \epsilon$. □

Example

Assume that the sequence (a_n) satisfies $|a_n - a_m| < \frac{1}{2^n}$ for all $m \geq n \geq 1$. Show that the sequence (a_n) is Cauchy.

Proof. Let $\epsilon > 0$. Choose an integer $N > 0$ such that $\frac{1}{2^N} < \epsilon$. Then for all integers $n, m > N$ with $m \geq n$, we have $|a_n - a_m| < \frac{1}{2^N} < \epsilon$. Thus (a_n) is Cauchy. □

Every Cauchy sequence is bounded

Clearly, the sequence $(1+(-1)^n)$ is not Cauchy, as two consecutive terms always differ by 2.

We are going to show that a sequence is Cauchy iff it converges. But let us first show a more basic property.

Lemma

Every Cauchy sequence (a_j) is bounded.

Proof. Let $\epsilon = 1 > 0$. There is an integer $N > 0$ such that $|a_j - a_k| < \epsilon = 1$ whenever $j, k > N$. Thus, if $j \geq N+1$, we have

$$\begin{aligned} |a_j| &\leq |a_{N+1} + (a_j - a_{N+1})| \\ &\leq |a_{N+1}| + |a_j - a_{N+1}| \\ &\leq |a_{N+1}| + 1 \equiv K. \end{aligned}$$

Let $L = \max\{|a_1|, |a_2|, \ldots, |a_N|\}$. If j is any natural number, then either $1 \leq j \leq N$, in which case $|a_j| \leq L$, or else $j > N$, in which case $|a_j| \leq K$. Set M = max{K,L}. Then, for any j, $|a_j| \leq M$ as required. $\square$

A sequence is Cauchy iff it converges

Theorem

Let $\{a_j\}$ be a sequence of real numbers. The sequence is Cauchy if and only if it converges to some limit α.

Proof. First assume that the sequence converges to a limit α. Let $\epsilon > 0$. Then there is an integer $N > 0$ such that if $j > N$ then $|a_j - \alpha| < \epsilon/2$. If $j, k > N$ then

$$|a_j - a_k| \leq |a_j - \alpha| + |\alpha - a_k| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

So the sequence is Cauchy.

Conversely, suppose that the sequence $\{a_j\}$ is Cauchy. Define

$$S = \{x \in \mathbb{R} : \text{there is a positive integer } k \text{ such that } x < a_j \text{ for all } j \geq k\}$$

Proof of theorem

By the lemma, the sequence $\{|a_j|\}$ is bounded above by some positive number M. Clearly, every number less than $-M$ is in S, so S is nonempty. Also S is bounded above by M. Thus $\sup S \in \mathbb{R}$. Let $\alpha = \sup S$. We claim that $\lim_{j\to\infty}\{a_j\} = \alpha$. To see this, let $\epsilon > 0$. Choose an integer $N > 0$ such that $|a_j - a_k| < \epsilon/2$ whenever $j, k > N$. Notice that this last inequality implies that

$$|a_j - a_{N+1}| < \epsilon/2 \quad \text{when} \quad j \geq N+1 \tag{2.12.1}$$

hence

$$a_j > a_{N+1} - \epsilon/2 \quad \text{when} \quad j \geq N+1.$$

Thus $a_{N+1} - \epsilon/2 \in S$ and it follows that

$$\alpha \geq a_{N+1} - \epsilon/2. \tag{2.12.2}$$

Proof of theorem

Line (2.12.1) also shows that

$$a_j < a_{N+1} + \epsilon/2 \quad \text{when} \quad j \geq N+1\,.$$

Thus $a_{N+1} + \epsilon/2 \notin S$. Note that every number smaller than a number in S is also in S. Thus every real number not in S is an upper bound of S. Thus,

$$\alpha \leq a_{N+1} + \epsilon/2. \tag{2.12.3}$$

Combining lines (2.12.2) and (2.12.3) gives

$$|\alpha - a_{N+1}| \leq \epsilon/2. \tag{2.12.4}$$

But then line (2.12.4) yields, for $j > N$, that

$$|\alpha - a_j| \leq |\alpha - a_{N+1}| + |a_{N+1} - a_j| < \epsilon/2 + \epsilon/2 = \epsilon\,.$$

This proves that the sequence $\{a_j\}$ converges to α, as claimed.

□

A complex sequence is Cauchy iff it converges

Corollary

Let $\{\alpha_j\}$ be a sequence of complex numbers. The sequence is Cauchy if and only if it is convergent.

Proof: Write $\alpha_j = a_j + ib_j$, with a_j, b_j are real.
Since $\max\{|a_i - a_k|, |b_j - b_k|\} \le |\alpha_j - \alpha_k|$ and
$|\alpha_j - \alpha_k| = |(a_j + ib_j) - (a_k + ib_k)| \le |a_i - a_k| + |b_j - b_k|$, we see that $\{\alpha_j\}$ is Cauchy if and only if $\{a_j\}$ and $\{b_j\}$ are Cauchy.
Also, $\{\alpha_j\}$ converges to a complex limit α if and only if $\{a_j\}$ converges to $\operatorname{Re}\alpha$ and $\{b_j\}$ converges to $\operatorname{Im}\alpha$.
These observations, together with the theorem, prove the corollary. □

Increasing, decreasing, monotone sequences

Definition

Let $\{a_j\}$ be a sequence of real numbers. The sequence is said to be *increasing* if $a_1 \leq a_2 \leq \ldots$. It is *decreasing* if $a_1 \geq a_2 \geq \ldots$. A sequence is said to be *monotone* if it is either increasing or decreasing.

We can show that a real sequence (a_n) is increasing (resp., decreasing) by checking that $a_{n+1} - a_n \geq 0$ (resp., $a_{n+1} - a_n \leq 0$) for all n.

For example, the sequence $(a_n) = (\frac{n}{n+2})$ is increasing since $a_{n+1} - a_n = \frac{n+1}{n+3} - \frac{n}{n+2} = \frac{2}{(n+2)(n+3)} > 0$ for all n. Similarly, we can show that $(\frac{3n+1}{n})$ is decreasing.

For a sequence (a_n) with all terms positive, we may show that (a_n) is increasing (resp., decreasing) by checking that $\frac{a_{n+1}}{a_n} \geq 1$ (resp., $\frac{a_{n+1}}{a_n} \leq 1$) for all n.

Convergence of bounded monotone sequences

For example, we can see that $(\frac{n!}{n^n})$ is decreasing by showing $\frac{a_{n+1}}{a_n} \leq 1$ for all n.

We say that a sequence (a_j) is *bounded above* (resp., *bounded below*) if there is a real number M such that $a_j \leq M$ (resp., $a_j \geq M$) for all j.

Theorem

If $\{a_j\}$ is an increasing sequence that is bounded above, then $\{a_j\}$ converges. If $\{b_j\}$ is a decreasing sequence that is bounded below, then $\{b_j\}$ converges.

Proof. Let $\{a_j\}$ be an increasing sequence that is bounded above. Let $\epsilon > 0$. Let $\alpha = \sup\{a_j : j \in \mathbb{N}\} \in \mathbb{R}$. Then $\alpha - \epsilon$ is not an upper bound for $\{a_j\}$. Hence, there is an integer N so that $\alpha - \epsilon < a_N$.

Then, if $\ell \geq N+1$, we have $\alpha - \epsilon < a_N \leq a_\ell \leq \alpha$ hence $|a_\ell - \alpha| < \epsilon$. Thus the sequence converges to α.
The proof for decreasing sequences is similar and is omitted. $\square$

Example

Let $a_1 = \sqrt{2}$ and set $a_{n+1} = \sqrt{2 + a_n}$ for all $n \geq 1$. Show that $\{a_n\}$ is increasing and bounded above, and find $\lim_{n\to\infty} a_n$.

Solution. Clearly, $a_2 = \sqrt{2+\sqrt{2}} > \sqrt{2} = a_1$. Assume that $a_{k+1} > a_k$ for some $k \in \mathbb{N}$. Then

$$a_{k+2} = \sqrt{2 + a_{k+1}} > \sqrt{2 + a_k} = a_{k+1}.$$

Thus $a_{n+1} > a_n$ for all n, so $\{a_n\}$ is increasing. It is easily shown by induction that $a_n < 3$ for all n. Thus $\{a_n\}$ converges to some number $\alpha > 0$. Taking the limits of both sides of $a_{n+1}^2 = 2 + a_n$ yields $\alpha^2 = 2 + \alpha$. Hence, $\alpha = 2$. $\square$

Properties of sup S and inf S

Corollary

Let S be a nonempty set of real numbers which is bounded above and below. Let β be its supremum and α its infimum. If $\epsilon > 0$ then there are $s, t \in S$ such that $|s - \beta| < \epsilon$ and $|t - \alpha| < \epsilon$.

Proof. Let $\epsilon > 0$. Since $\alpha = \inf S$ is the greatest lower bound of S, $\alpha + \epsilon$ is not a lower bound for S. Thus there is an element $t \in S$ such that $t < \alpha + \epsilon$. Of course, we also have $\alpha \leq t$. Hence $\alpha \leq t < \alpha + \epsilon$. Therefore, $|t - \alpha| < \epsilon$.
The claim about β and s can be shown similarly. □

For example, if $S = \{1/n : n \in \mathbb{N}\}$, then $\alpha = \inf S = 0$ and $\beta = \sup S = 1$, and for each $\epsilon > 0$, one can find $s, t \in S$ such that $|s - \beta| < \epsilon$ and $|t - \alpha| < \epsilon$. Indeed, one may choose $s = 1$ and $t = 1/n$ for some $n > 1/\epsilon$.

Sequences that are not Cauchy

As a sequence diverges if and only if (iff) it is not Cauchy, it is helpful to have a clear understanding of sequences that are not Cauchy.

Fact

A sequence (a_n) is not Cauchy iff there is an $\epsilon > 0$ such that for each $N > 0$, there are integers j and k such that $j, k > N$ and $|a_j - a_k| \geq \epsilon$.

In particular, this idea may be used to give a proof (by contradiction) to problem 12 in the Exercises.

The Pinching Principle

We conclude the section with a result that is very useful for calculating the limits of certain sequences.

Proposition (The Pinching Principle)

Let $\{a_j\}, \{b_j\}$, and $\{c_j\}$ be sequences of real numbers satisfying

$$a_j \le b_j \le c_j$$

for every j sufficiently large. If $\lim_{j\to\infty} a_j = \lim_{j\to\infty} c_j = \alpha$ *for some real number α, then* $\lim_{j\to\infty} b_j = \alpha$.

Proof. The proof is left as an exercise.
Hint: $\alpha - \epsilon < a_j \le b_j \le c_j < \alpha + \epsilon$ ensures $|b_j - \alpha| < \epsilon$. □

An application of the Pinching Principle

To apply the Pinching Principle to a sequence $\{b_j\}$, one has to consider desirable auxiliary sequences $\{a_j\}$ and $\{c_j\}$ whose limits are easy to compute.

Observe that for any sequence $\{b_j\}$, $\lim_{j\to\infty} b_j = 0$ iff $\lim_{j\to\infty} |b_j| = 0$.

Example

Find $\lim_{j\to\infty} \cos(j^2)/j$.

Solution. Note that $0 \le |\cos(j^2)/j| \le 1/j$ for all j, and $\lim_{j\to\infty} 0 = \lim_{j\to\infty} 1/j = 0$. By the Pinching Principle, $\lim_{j\to\infty} |\cos(j^2)/j| = 0$. Hence, $\lim_{j\to\infty} \cos(j^2)/j = 0$. □