Math 4111
September 23, 2020 Lecture

Steven G. Krantz

September 7, 2020



Figure: This is your instructor.

2.1 Convergence of Sequences

Homework: 1, 2, 3, 8, 10, 11, 12.

A *sequence* of real numbers is a function $\phi : \mathbb{N} \to \mathbb{R}$. Similarly, a sequence of complex numbers is a function $\phi : \mathbb{N} \to \mathbb{C}$.

We often denote a sequence $a : \mathbb{N} \to \mathbb{R}$ as $a_1, a_2, \ldots,$ or $\{a_n\}_{n=1}^{\infty}$, or $(a_n)_{n=1}^{\infty}$, or simply as $\{a_n\}$ or $\{a_j\}$.

Recall the familiar sequence $\{(1 + 1/n)^n\}$ from calculus.

$(\cos \frac{n\pi}{3} + i \sin \frac{n\pi}{3})$ is a sequence of complex numbers.

However, a sequence may not have a clear pattern or formula. For example, consider the sequence of decimal digits of π: $3, 1, 4, 1, 5, 9, 2, 6, 5, 3, 5, 8, 9, 7, 9, 3, 2, 3, 8, 4, 6, 2, 6, 4, 3, 3, 8, \ldots$.

Limit of a Sequence

If the terms of a sequence (a_n) approach a fixed number as n goes to infinity, we say the sequence has a limit. This important concept is is made precise as follows.

Definition (Limit of a Sequence)

A sequence $\{a_n\}$ of real (or complex) numbers *converges to* a real (or complex) number c if, for each real number $\epsilon > 0$, there is a positive integer N such that for all integers n with $n > N$, we have $|a_n - c| < \epsilon$. In this case, we say that c is the *limit* of the sequence $\{a_n\}$ and that $\{a_n\}$ is *convergent*, and we write $\lim_{n\to\infty} a_n = c$, or $\lim_{n\to\infty} a_n = c$, or $a_n \to c$ (as $n \to \infty$).

If a sequence does not converge to any number, we say the sequence *diverges* or is *divergent*.

Note that in the above definition, N depends on ϵ. Generally speaking, a smaller $\epsilon > 0$ may require a bigger N.

Informally, we may say that $\lim_{n\to\infty} a_n = c$ means a_n can be made arbitrarily close to c as long as n is sufficiently large.

Example

Show that $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$.

Proof: Let $\epsilon > 0$. Observe that $|\frac{1}{\sqrt{n}} - 0| < \epsilon$ if and only if $n > \frac{1}{\epsilon^2}$. Let $N = \lfloor \frac{1}{\epsilon^2} \rfloor + 1$. Then for all integers n with $n > N$, we have $|\frac{1}{\sqrt{n}} - 0| < \epsilon$. Therefore, $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$. $\square$

Boundedness

Definition

We say a sequence $\{a_n\}$ is *bounded* if there is a number $M > 0$ such that $|a_n| \leq M$ for all n.

Example

Show that the sequence $(a_n) = ((-1)^n + 1)$ is bounded, but divergent.

Proof: Since $|a_n| \leq 2$ for all n, the sequence (a_n) is bounded. Assume that (a_n) converges to a number c. For $\epsilon = 1/2$, there is an integer $N > 0$ such that $|a_n - c| < \epsilon = 1/2$ for all $n > N$. Let n be an even integer with $n > N$. Then

$2 = |2 - 0| = |a_n - a_{n+1}| = |a_n - c + c - a_{n+1}|$

$\leq |a_n - c| + |c - a_{n+1}| < \epsilon + \epsilon = 1/2 + 1/2 = 1,$

which says $2 < 1$, a contradiction. Thus (a_n) diverges. $\square$

Uniqueness of the limit; Boundedness

Theorem

Let $\{a_n\}$ be a convergent sequence. Then the limit of the sequence $\{a_n\}$ is unique and the sequence is bounded.

Proof: Suppose that the sequence $\{a_n\}$ has two limits c and d. Let $\epsilon > 0$. Then there is a positive integer N_1 such that for all $n > N_1$, $|a_n - c| < \epsilon$. Similarly, there is a positive integer N_2 such that for all $n > N_2$, $|a_n - d| < \epsilon$. Let j be an integer with $j > \max\{N_1, N_2\}$. Then $|c - d| = |c - a_j + a_j - d|$ $\leq |c - a_j| + |a_j - d| < \epsilon + \epsilon = 2\epsilon$. As $|c - d| < 2\epsilon$ for every $\epsilon > 0$, we get $c = d$. Thus the limit is unique.

To show that $\{a_n\}$ is bounded, we let $c = \lim_{n\to\infty} a_n$ and $\epsilon = 1$. Then there is a positive integer N such that for all $n > N$, we have $|a_n - c| < 1$, and hence,
$|a_n| = |a_n - c + c| \leq |a_n - c| + |c| < 1 + |c|$.
Let $M = \max\{|a_1|, \ldots, |a_N|, 1 + |c|\}$. Then $|a_n| \leq M$ for all $n \in \mathbb{N}$. $\square$

Next, we consider several fundamental properties of limits involving arithmetic operations on sequences. Such properties are powerful tools for computing complicated limits.

Some basic properties of limits

Theorem

Suppose that $\lim_{j\to\infty}\{a_j\} = \alpha$ *and* $\lim_{j\to\infty}\{b_j\} = \beta$, *where* $\alpha, \beta \in \mathbb{C}$. *Then we have:*

(1) *For any constant* c, $\lim_{j\to\infty}(c \cdot a_j) = c \cdot \alpha$;

(2) $\lim_{j\to\infty}(a_j + b_j) = \alpha + \beta$;

(3) $\lim_{j\to\infty}(a_j \cdot b_j) = \alpha \cdot \beta$;

(4) *If* $b_j \neq 0$ *for all* j *and* $\beta \neq 0$, *then* $\lim_{j\to\infty}(a_j/b_j) = \alpha/\beta$.

Proof: To prove **(1)**, we may assume that $c \neq 0$ (for when $c = 0$ there is nothing to prove). Let $\epsilon > 0$. Choose an integer $N > 0$ such that for all $j > N$ it holds that $|a_j - \alpha| < \frac{\epsilon}{|c|}$. For such j we have that $|c \cdot a_j - c \cdot \alpha| = |c| \cdot |a_j - \alpha| < |c| \cdot \frac{\epsilon}{|c|} = \epsilon$. This proves the first assertion.

Proofs of some basic limit properties

We now prove **(2)**. Let $\epsilon > 0$. Because $\lim_{j\to\infty} a_j = \alpha$, there is an integer $N_1 > 0$ such that $|a_j - \alpha| < \frac{\epsilon}{2}$ for all $j > N_1$. Similarly, there is an integer $N_2 > 0$ such that $|b_j - \beta| < \frac{\epsilon}{2}$ for all $j > N_2$. Let $N = \max\{N_1, N_2\}$. For for all $j > N$, we have

$$\begin{aligned} |(a_j + b_j) - (\alpha + \beta)| &= |(a_j - \alpha) + (b_j - \beta)| \\ &\leq |a_j - \alpha| + |b_j - \beta| \\ &< \epsilon/2 + \epsilon/2 = \epsilon\,. \end{aligned}$$

Thus **(2)** holds.

To prove **(3)**, note that the sequence $\{a_j\}$ is bounded (by the last theorem): for some positive number M, $|a_j| < M$ for every j. Let $\epsilon > 0$. Choose an integer $N > 0$ so that $|a_j - \alpha| < \epsilon/(2M + 2|\beta|)$ when $j > N$. Also choose an integer $\widetilde{N} > 0$ such that $|b_j - \beta| < \epsilon/(2M + 2|\beta|)$ when $j > \widetilde{N}$.

Proofs of some basic limit properties

Then, for all $j > \max\{N, \widetilde{N}\}$, we have

$$\begin{aligned}
|a_j b_j - \alpha\beta| &= |a_j b_j - a_j\beta + a_j\beta - \alpha\beta)| \\
&= |a_j(b_j - \beta) + (a_j - \alpha)\beta| \\
&\leq |a_j(b_j - \beta)| + |\beta(a_j - \alpha)| \\
&< M \cdot \frac{\epsilon}{2M + 2|\beta|} + |\beta| \cdot \frac{\epsilon}{2M + 2|\beta|} \\
&\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon.
\end{aligned}$$

So the sequence $\{a_j b_j\}$ converges to $\alpha\beta$. Thus **(3)** holds.

Proofs of some basic limit properties

In view of **(3)**, to prove **(4)** it suffices to prove that $\lim_{j\to\infty}(1/b_j) = 1/\beta$, under the assumptions that $b_j \neq 0$ for all j and $\beta \neq 0$. Note that for any two complex numbers x and y, by the triangle inequality, we have $|x| = |x + y - y| \leq |x + y| + |y|$, and hence, $|x + y| \geq |x| - |y|$. Since $\lim_{j\to\infty} b_j = \beta \neq 0$, there is an integer N_1 such that for all $j > N_1$, we have $|b_j - \beta| < |\beta|/2$ and hence, $|b_j| = |\beta + b_j - \beta| \geq |\beta| - |b_j - \beta| > |\beta|/2$. Let $\epsilon > 0$. There is an integer N_2 such that $|b_j - \beta| < \frac{|\beta|^2}{2}\epsilon$ for all $j > N_2$. It can be seen that for all $j > \max\{N_1, N_2\}$, we have $|\frac{1}{b_j} - \frac{1}{\beta}| = \frac{|b_j-\beta|}{|\beta b_j|} < \epsilon$. Thus, $\lim_{j\to\infty}(1/b_j) = 1/\beta$. $\square$